Structural Properties in δ-Hyperbolic Networks: Algorithmic Analysis and Implications

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ABSTRACT

In graph theory, the δ-hyperbolicity is a global property that shows how close a given graph's structure is to the tree's structure metrically. It embeds multiple properties that facilitate solving several problems that found to be hard in the general graph form. Interestingly, not only that δ-hyperbolicity provides an idea about the structure of the graph, but also it explains how information navigates throughout the network. Therefore, δ-hyperbolicity has several applications in diverse applied fields. My PhD dissertation focuses on analyzing and exploiting structural properties of hyperbolic networks for different applications.

Keywords

δ-Hyperbolic graph; core-periphery structure; vertex eccentricity; eccentricity centrality; shortest paths.

1. INTRODUCTION

Using networks to describe systems that are composed of elements and their interactions aids analyzing and understanding them. Therefore, networks in multiple disciplines ranging from computer science and engineering to physics and systems biology are being modeled as graphs. Using graph-theoretical tools for analyzing complex networks to characterize their structures has been the subject of much research. Knowing that a given graph does not have a completely random structure improves the general understanding of its different aspects. Therefore, identifying any structural properties that a graph may possess could indeed facilitate analyzing it.

The core-periphery structure has been widely recognized in many network disciplines. It suggests partitioning the graph into two parts: the core which is dense and cohesive and the periphery which is sparse and disconnected. Vertices in the periphery part interact through a series of intermediate core vertices. This pattern of communication (where traffic tends to concentrate on a specific subset of the vertices) has been observed in trees where distant nodes communicate via the central node (or nodes) in the tree. δ-Hyperbolicity, which is a measure that shows how close a graph is to a tree, suggests that any shortest path between any pair of vertices bends (to some extent) towards the core of the graph. This phenomenon has been justified by the global curvature of the network which (in case of graphs) can be measured using hyperbolicity (sometimes called also the negative curvature) [12]. Multiple complex networks such as the Internet [15, 6], data networks at the IP layer [12], and social and biological networks [2] show low δ-hyperbolicity (low hyperbolicity suggests a structure that is close to a tree structure). Also, it has been observed that networks with this property have highly connected cores [12]. Generally, the core of a graph is specified according to one or more centrality measures such as the degree, the betweenness, and the eccentricity centrality measures.

1.1 The problem

The δ-hyperbolicity of graphs embeds multiple properties that facilitate solving several problems that found to be difficult in the general graph form; for example, diameter estimation [4], distance and routing labeling [5], and several routing problems [10]. My PhD dissertation focuses on analyzing and exploiting structural properties of real-world networks for various applications. Specifically, analyzing vertex eccentricities in δ-hyperbolic graphs. In [3], we algorithmically analyze the δ-hyperbolicity property and we exploit it to partition a graph into core and periphery parts. To achieve this, we formalize the notion of the eccentricity layering of a graph and employ it to introduce a new property that we find intrinsic to hyperbolic graphs: *the eccentricity-based bending property*. Moreover, we investigate the essence of the bending in shortest paths by studying its relationship to the distance between vertex pairs. This part is summarized in Section 4. Our current work involves investigating another structural property of graphs that depends on the unimodality of the eccentricity function (Section 5). We also analyze using this property to identify the core vertices in a given network (Section 5.1).

2. RELATED WORK

In the study of communication networks, the core is usually identified by the small dense part that carries out most traffic under shortest path routing [12]. It is quite natural to associate the concepts of the network's core and its center. The notion behind centrality is identifying vertices that are high contributors. There are multiple centrality measures in the literature. The betweenness centrality expresses how

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much effect each vertex has in the communication. The eccentricity centrality suggests that the center of the graph includes the vertices that have the shortest distance to all other vertices. In [14], Seidman proposes the k-core decomposition as a tool to study the structural properties of large networks focusing on subsets of increasing degree centrality. It partitions the graph into subsets each of which is identified by removing vertices of degree smaller than k. Holme [9] introduces a coefficient that measures if a network has a clear core-periphery structure based on the closeness centrality. Leskovec et.al. [11] study community structures in large networks. They identify the existence of multiple (smaller) communities that are attached to the core of the network with very few connections. They also observe that some graphs have cores with a nested core-periphery structure.

3. PRELIMINARIES ON GRAPH THEORY

A simple undirected graph $G = (V, E)$ naturally defines a metric space (V, d) on its vertex set V. The distance $d(u, v)$ is defined as the number of edges in a shortest path $\rho(u, v)$ that connects two vertices u and v. The *diameter* of the graph $diam(G)$ is the length of the longest shortest path between any two vertices u and v in the graph, i.e., $diam(G) = \max_{u,v \in V}\{d(u, v)\}$. Obviously, when the graph is disconnected, the value of the diameter is undefined. The *eccentricity* of a vertex u is $ecc(u) = \max_{v \in V}\{d(u, v)\}$, i.e., the distance between u and any of its farthest neighbors v. The minimum value of the eccentricity represents the graph's *radius*: $rad(G) = min_{u \in V}\{ecc(u)\}$. The *center* of the graph is defined as $C(G) = \{u \in V : ecc(u) = rad(G)\}$. For each integer $r \geq 0$, let $N_r(u)$ denotes the *neighborhood* of distance at most r centered at u, i.e., $N_r(u) = \{v \in V : d(u, v) \leq r\}$. Accordingly, $N_1(u)$ has all vertices that are adjacent to u.

δ-Hyperbolicity.

In smooth geometry, hyperbolicity captures the notion of negative curvature which can be generalized as δ-hyperbolicity in more abstract concept of metric spaces including graphs. The presence of hyperbolic networks in a variety of applications attracted researchers to investigate the negative curvature of different types of graphs. The δ-hyperbolicity measure of a metric space measures how close the metric structure is to the tree structure [8]. There are multiple equivalent definitions (up to constant factors [4]) for hyperbolicity. In our work, we use the following four-point condition definition. Given a graph $G = (V, E)$, x, y, u, and $v \in V$ are four distinct vertices, and the three sums: $d(x, y) + d(u, v)$, $d(x, u) + d(y, v)$, and $d(x, v) + d(y, u)$ sorted in a non-increasing order, the hyperbolicity of the quadruple x, y, u, v denoted as $\delta(x, y, u, v)$ is defined as: $\delta(x, y, u, v) = ((d(x, y) + d(u, v)) - (d(x, u) + d(y, v)))/2$. The δ-hyperbolicity of the graph G denoted as $\delta(G)$ (or simply δ) is $\delta(G) = \max_{x,y,u,v \in G} \delta(x, y, u, v)$. Trees are 0-hyperbolic ($\delta(G) = 0$), cliques are also 0-hyperbolic, and chordal graphs are at most 1-hyperbolic. Generally, the smaller the value of δ the closer the graph is to a tree (metrically).

Eccentricity layering of a graph.

The *eccentricity layering* of a graph $G = (V, E)$ ($\mathcal{EL}(G)$) partitions its vertices into concentric circles or layers $\ell_r(G)$, $r = 0, 1, \ldots$. Each layer r is defined as $\ell_r(G) = \{u \in V : ecc(u) - rad(G) = r\}$. Here r represents the index of the layer. The inner-most layer (layer 0) encloses the graph's center $C(G)$; this layer has index $r = 0$. Then the first layer includes all vertices who have their eccentricities equal to $rad(G) + 1$, and so on. The vertices in the last layer have eccentricities equal to the diameter of the graph. See Figure 1. Any vertex $v \in \ell_r(G)$ has *level* (or layer) $level(v) = r$.

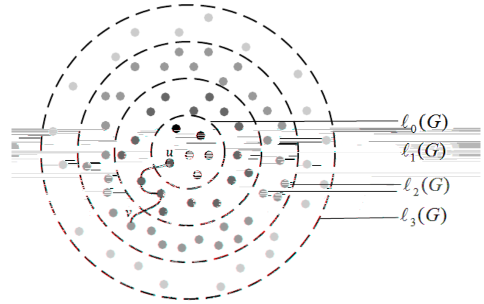

Figure 1: Eccentricity Layering of a graph.

4. EARLIER WORK

It was suggested in [12] that the highly congested cores in many communication networks can be due to their negative curvature. Those cores are represented by vertices that belong to most shortest paths and (or) have minimum distances to all other vertices. It was also observed that the negative curvature causes most of the shortest paths to bend making the peak of the arc formed by a shortest path to pass through the core. In [3], we formalized this notion of bending in shortest paths by introducing an important property that is intrinsic to δ-hyperbolic graphs (the eccentricity-based bending property). Then we used this property to partition a graph into core and periphery parts.

4.1 Eccentricity-Based Bending Property of δ-Hyperbolic Networks

Let $G = (V, E)$ be a δ-hyperbolic graph, $\mathcal{EL}(G)$ be its eccentricity layering, and $C(G)$ be its center. In [4], the following property of δ-hyperbolic graphs was proven.

Lemma 1[4]. *Let G be a δ-hyperbolic graph and x, y, v, u be its four arbitrary vertices. If $d(v, u) \geq \max\{d(y, u), d(x, u)\}$, then $d(x, y) \leq \max\{d(v, x),\ d(v, y)\} + 2\delta$.*

We use this property to establish the following.

Proposition 1. *Let G be a δ-hyperbolic graph and x, y, s be arbitrary vertices of G. If $d(x, y) > 4\delta + 1$, then $d(w, s) < \max\{d(x, s), d(y, s)\}$ for any middle vertex w of any shortest (x, y)-path.*

Proposition 2. *Let G be a δ-hyperbolic graph and x, y be arbitrary vertices of G. If $d(x, y) > 4\delta + 1$, then on any shortest (x, y)-path there is a vertex w with $ecc(w) < \max\{ecc(x), ecc(y)\}$.*

We define the bend in shortest paths between two distinct vertices u and v with $d(u, v) \geq 2$ as follows: $\forall\ u, v \in V$ $bend(u, v) = \min\{level(z) : z \in V$ and $d(u, z) + d(z, v) = d(u, v)\}$. Here $level(z) = r$ iff $z \in \ell_r(G)$. We say that shortest paths between u and v bend if and only if a vertex z with $ecc(z) < \max\{ecc(u), ecc(v)\}$ exists in a shortest path between them. The parameter *bend* decides the extent (or the level) to which shortest paths between vertices u and v curve towards the center (since we are always looking for a z that belongs to a smaller layer according to the eccentricity layering). Note that in some cases $bend(u, v)$ will be assigned either $ecc(u)$ or $ecc(v)$, whatever is smaller. For example, see the shortest path $\rho(u, v)$ in Figure 1.

4.2 Core Identification Using the Eccentricity-Based Bending Property

A well-de ned center of a graph is a good starting point for locating its core. According to the pattern of data exchange discussed earlier, we identify the core using the eccentricity centrality measure. Even though the center contains all vertices that are closer to other vertices, this subset is not su cient. More vertices should be added to the core according to their participation in routing the tra c. We decide the participation of each vertex based on its eccentricity and whether or not it lies on a shortest path between a vertex pair. Next we discuss two core-periphery separation models.

The Maximum-Peak Model. This model identi es a separation layer index $p \geq 0$ and de nes the core as the subset of vertices formed by layers $\ell_0(G), \ell_1(G), ..., \ell_p(G)$. In light of the eccentricity-based bending property, each $bend(x, y)$ for a pair of vertices x and y represents a *peak* for $\rho(x, y)$. Here we are locating the index of the lowest layer p over all layers that vertex pairs bend to. Index p represents the separation point where the layers can be partitioned to a core and a periphery. After identifying all peaks, the core will be de ned as $core(G) = \bigcup_{r=0}^{p} \ell_r(G)$. Since the core according to this model is de ned using the layers, some vertices may be added to the core that do not have real contribution (they were added only for their location). This issue was resolved in the second model.

The Minimum Cover Set Model. This model starts with the core as an empty set and expand it to include vertices which have smaller eccentricity, are closer to the center, and participate in the tra c. This expansion should be orderly, rst incorporating the vertices that have higher priority, and then vertices who are less eligible. For each vertex $v \in V$, we de ne three parameters according to which we prioritize the vertices.

- The *eccentricity* $ecc(v)$. Vertices with smaller eccentricities have higher priority to be in the graph's core.
- The *distance-to-center* $f(v) = d(v, C(G))$. Vertices with small $f(v)$ have higher priority of being in the core.
- The *betweenness* $b(v)$ which measures how many pairs of distant vertices x and y have v in one of their shortest paths (versus all shortest paths in the classic de nition of the betweennes). It quanti es the participation of a vertex v in the tra c ow process, and we de ne it as: $b(v)$ = number of pairs $x, y \in V$ with $v \neq x, v \neq y, d(x, y) \geq 2$ and $d(x, v) + d(v, y) = d(x, y)$. According to the core-periphery organization, the betweenness of a vertex should increase as its eccentricity decreases.

Our goal is to identify the smallest subset of vertices that participate in all tra c throughout the network. The algorithm for this model comprises two stages. First, in a priority list T we lexicographically sort the vertices according to the three attributes: $ecc(v)$, $f(v)$, and $b(v)$. T now has the vertices in the order that they should be considered to become part of the core. The goal is to ensure that there exists at least one vertex $v \in core(G)$ such that $v \in \rho(x, y)$ for each pair of vertices $x, y \in V$. In such case, we say that a shortest path $\rho(x, y)$ is *covered by* v. The second stage starts with a vertex v at the head of T being removed from T and added to an initially empty set $core(G)$. This vertex must cover at least one pair. After this initial step, the process continues by repeatedly removing the vertex v at the head of T and adding it to $core(G)$ if and only if v covers an uncovered yet pair x and y (when there is at least one vertex $v \in core(G)$ that covers a pair (x, y), then it becomes covered). This step should run until all pairs are covered. Now the vertices in set $core(G)$ represent the core of the graph while the remaining vertices represent the periphery.

We applied both models to a wide set of biological networks. For more details, the reader is referred to [3].

5. ONGOING RESEARCH

Even though vertices with smaller eccentricities are more likely to be in the core (the eccentricity based property), not all vertices with equal eccentricity values have the same quality. Our current work involves investigating the unimodality of the eccentricity function in graphs and we analyze using this property to identify the core vertices in a given network.

Let $G = (V, E)$ be an unweighted graph, $\mathcal{EL}(G)$ be its eccentricity layering, $C(G)$ be its center, and $ecc(u)$ be the eccentricity function de ned for every vertex $u \in V$. The eccentricity function of G is *unimodal* when every non central vertex u has a neighbor vertex v with less eccentricity. Otherwise, the eccentricity function of G is *non-unimodal*.

De nition 1 [7]. Given a graph $G = (V, E)$, its eccentricity function $ecc(u) = \max_{v 2 V}\{d(u, v)\}$ is unimodal if for every vertex $u \in V \setminus C(G)$ there exists at least one vertex $v \in N_1(u)$ such that $ecc(u) = ecc(v) + 1$.

Figure 2(a) shows a graph with a unimodal eccentricity function. When the eccentricity function of a graph is unimodal, the relationship between the eccentricity of a vertex $u \in V$, its distance to the graph's center, and the radius of the graph can be described as follows. $\forall u \in V$, $ecc(u) = d(u, c) + rad(G)$ [7]. The name unimodal is due to the fact that the local minimum (vertex with local minimum eccentricity) and the global minimum (the graph's center) coincide. Starting at each vertex u and moving towards the neighbor with less eccentricity (which always exists due to the unimodality), we get to the center in $d(u, c)$ hops, where $c \in C(G)$ and it is the closest to u. According to the graph's eccentricity function, those paths that connect every non central vertex to the center are all monotonically decreasing. Now we generalize the discussion to include graphs with non-unimodal eccentricity functions. We concentrate on the paths (possibly and preferably but not necessarily the shortest) that connect each vertex with the center $C(G)$ since we are investigating the unimodality of the eccentricity function.

De nition 2. Let $c \in C(G)$ be a closest vertex in the graph's center to u and $p(u, c) = v_1, v_2, ..., v_m$, where $v_1 = u$ and $v_m = c$, be a path between vertices u and c. The path $p(u, c)$ can be

1. *Monotonically decreasing* if $ecc(v_i) > ecc(v_{i+1})$ $\forall$ $v_i \in p(u, c)$, $1 \leq i \leq d(u, c) - 1$. In this case, we say that vertex u has a monotonically decreasing path to $C(G)$.
2. *Monotonically non-increasing* if $ecc(v_i) \geq ecc(v_{i+1})$ $\forall$ $v_i \in p(u, c)$, $1 \leq i \leq d(u, c) - 1$. If such a path exists and no monotonically decreasing path to the center

exists, then we say that the vertex has a monotonically non-increasing path to the center.

3. $Non\text{-}monotonic$ if $\exists v_i \in p(u,c)$ such that $ecc(v_i) < ecc(v_{i+1})$ where $1 \leq i \leq d(u,c) - 1$. A vertex has a non-monotonic path to the center if no other monotonically decreasing or monotonically non-increasing paths to the center exist.

In Figure 2(c), the path $p(B,E)$, which connects vertex B to $C(G) = \{E\}$ is monotonically decreasing, path $p(G,E)$ is monotonically non-increasing, and path $p(A,E)$ is a non-monotonic path. One way to achieve this categorization is by creating the *eccentricity directed graph*. The eccentricity directed graph of a graph $G = (V,E)$ is the graph $\overrightarrow{G} = (V, E^{0})$ such that an edge $e_{uv} \in E^{0}$ if $ecc(u) < ecc(v)$ (the eccentricities of u and v with respect to graph G). In the case where $ecc(u) = ecc(v)$, the edge e_{uv} will be bidirectional in $\overrightarrow{G}$. Algorithm 1 shows how $\overrightarrow{G}$ can be used to categorize the vertices based on the monotonicity of their paths to the graph's center. Obviously, determining whether or not a vertex has a monotonically decreasing path to the center is straight forward. All we need is to compare the level (layer) of the vertex with its distance to the center. The problem of determining if a vertex has a non-increasing monotonic path or a non-monotonic path can be solved as a reachability problem.

The relationship that connects the eccentricity of a vertex, its distance to the graph's center, and the radius of the graph provided earlier can be generalized as follows.

Lemma 2. Let $G = (V,E)$ be an unweighted undirected graph with the radius $rad(G)$. The inequality $ecc(u) \leq d(u, C(G)) + rad(G)$ is true for every vertex $u \in V$.

According to those variations, we de ne the locality of a non central vertex u as $loc(u) = \min\{d(u,v) : v \in V$ and $ecc(u) = ecc(v) + 1\}$. The locality decides the number of hops between u and its closest neighbor v such that $ecc(v) = ecc(u) - 1$. If G has a unimodal eccentricity function, then every non-central vertex $u \in V$ has $loc(u) = 1$. However, if such v does not exist in $N_1(u)$ (which may be the case if G has a non-unimodal eccentricity function), a vertex with less eccentricity may exist two or more hops away. For example, consider vertex A in Figures 2(b) and 2(c). We de ne the locality of any $u \in C(G)$ as $loc(u) = 1$ since a vertex with less eccentricity does not exist. Based on this description, it is obvious that the eccentricity function of a tree is always unimodal. Also, for some graphs, it has been shown that vertices with locality > 1 only exist in lower layers according to the eccentricity layering of a graph (layers that are close to the graph's center) [13].

5.1 Vertex Locality and the Graph's Core

If the problem in hand is about locating the site that minimizes the distance to every other location in the network, then the problem becomes a centrality problem. A solution would be by identifying the core with the minimum eccentricity centrality. In [3], we started the core set as the vertices with the minimum eccentricity, then we expanded this core by adding other vertices with smaller eccentricities and with high betweenness centrality. In this section, we further investigate the nature of those core vertices with respect to their locality. In graphs with unimodal eccentricity functions, every local minimum is a global minimum. In contrast, in graphs where the eccentricity function is non-unimodal, some non central vertices are local minimums.

Algorithm 1. Categorizing vertices based on the monotonicity of their paths to the graph's center. For a vertex u, c is a vertex in $C(G)$ closest to u and $level(u) = r$ if $u \in \ell_r(G)$.

Input: An eccentricity directed graph $\overrightarrow{G} = (V, E)$.
Output: V_1: set of vertices with monotonically decreasing paths, V_2: set of vertices with monotonically non-increasing paths, and V_3: set of vertices with non-monotonic paths.

$V_1 \leftarrow \emptyset$, $V_2 \leftarrow \emptyset$, $V_3 \leftarrow \emptyset$
for each $u \in V$ **do**
  if $level(u) = d(u,c)$ **then**
    $V_1 \leftarrow V_1 \cup u$
  else
    if $d(u,c) \neq \emptyset$ **then**
      $V_2 \leftarrow V_2 \cup u$
    else
      $V_3 \leftarrow V_3 \cup u$
return V_1, V_2, V_3

This set of vertices is represented by vertice of higher localities (locality > 1). This motivates investigating whether those non local vertices represent cores for a di erent set of vertices.

Consider a graph $G = (V,E)$ and two non central and non adjacent vertices u and v, a shortest path $\rho(u,v)$ either bends to the center of the graph or does not bend. As shown in [3], this bend is highly a ected by its length. Longer shortest paths tend to bend more than shorter shortest paths. It is natural to think that for some pairs of vertices with equal eccentricities and who have a relatively short shortest path between them to not bend to the center of the graph. For example, consider $\rho(G,H)$ that does not bend and $\rho(I,K)$ that only bends to $\ell(G)_1$ in Figure 2(c).

What we plan to do next is to use the locality of the vertices and the eccentricity centrality to identify the core vertices. Given a graph, we de ne its core as

$$core(G) = C(G) \cup \{u : loc(u) > 1\}.$$

The idea is that vertices in the core can be partitioned into two subsets. The rst subset has vertices that are on shortest paths between bending vertex pairs (vertex pair (x,y) that has a vertex u on a shortest path between x and y and $ecc(u) < \max\{ecc(x), ecc(y)\}$). Those vertices have small localities. The second subset has vertices that are on a shortest path between vertex pairs that do not bend to the center of the graph. Those vertices are of a higher locality (of locality greater than one).

As we observed when trying this algorithm on a set of real-world networks, vertices in the graph's center and vertices with locality > 1 have the following two properties in common. (1) They both represent local minimums according to their eccentricities. (2) Generally, they have higher betweenness values (when compared with other vertices in the graph).

We applied this idea on a wide set of networks. Take a look at Table 1 for a sample of three Autonomous System graphs. The table shows the percent of vertices with a higher locality (loc > 1) in each graph. They do not represent more than 3% of vertices in the overall graph and they are located in lower layers (close to the center). Table 1 also shows the percent of vertices in the core de ned as $core(G) = C(G) \cup \{u : loc(u) > 1\}$ and the percent of shortest paths covered by at least one vertex in this core.

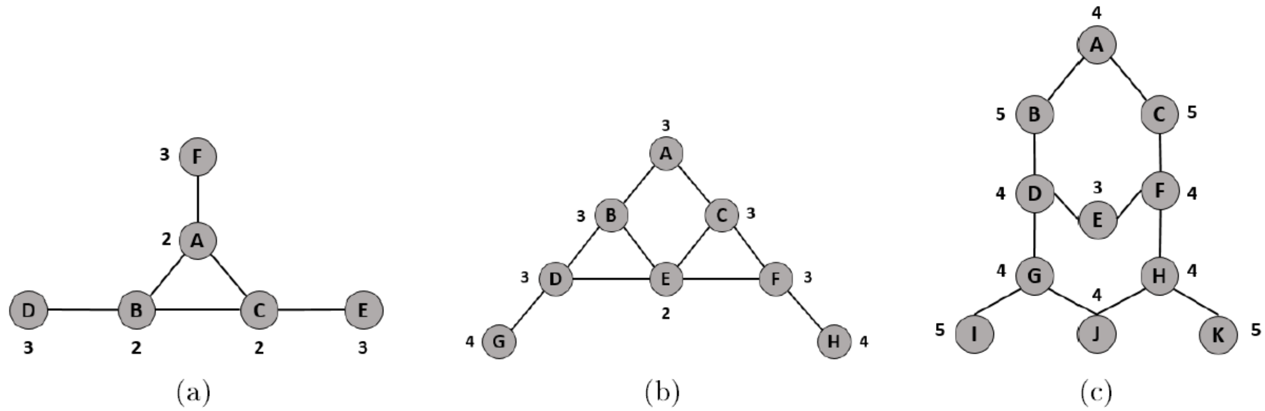

Figure 2: Three graphs with di erent eccentricity functions. The number next to each vertex indicates its eccentricity. (a) A graph with a unimodal eccentricity function. (b) A graph with a non-unimodal eccentricity function. (c) A graph with a non-unimodal eccentricity function.

Table 1: Results on Internet graphs **(Networks are available at[1]).**

Network	$\|V\|$	$diam(G)$	% of vertices with loc > 1	% of vertices in $core(G)$	% of shortest paths covered by $core(G)$
ASG-97	3015	9	1%	2%	$\approx$ 94%
ASG-499	4885	11	2%	13%	$\approx$ 99%
ASG-799	5357	9	2.7%	2.3%	$\approx$ 80%

6. CONCLUSIONS AND FUTURE WORK

In this paper, two structural properties of hyperbolic networks were studied: the eccentricity-based bending property and the unimodality of the eccentricity function. Moreover, using both properties to identify the core vertices of a given graph have been investigated. In [3], we have studied the relationship between the hyperbolicity of a graph and the conciseness of its core. We nd that the more hyperbolic the graph is (i.e., the closer its structure is to a tree's structure), the more it exhibits a clear-cut core periphery structure. For the next stage, we plan to plan to continue investigating the relationship between the δ-hyperbolicity of a graph and the unimodality of its eccentricity function. We also plan to investigate identifying central paths in a graph using the locality of vertices.

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