KUNGL TEKNISKA HÖGSKOLAN
Royal Institute of Technology
Numerical Analysis and Computing Science

TRITA-NA-D9708 • CID-17, KTH, Stockholm, Sweden 1997

# The Garden of Knowledge as a Knowledge Manifold

## A Conceptual Framework for Computer Supported Subjective Education

Ambjörn Naeve

CID
Centre for
User Oriented IT Design

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**Report number:** TRITA-NA-D9708, CID-17
**ISSN number:** ISSN 1403-073X
**Publication date:** September 1997
**E-mail of author:** amb@nada.kth.se
**URL of author:** http://www.nada.kth.se/~ambjorn

**Reports can be ordered from:**
CID, Centre for User Oriented IT Design
Nada, Dept. Computing Science
KTH, Royal Institute of Technology
S-100 44 Stockhom, Sweden
telephone: + 46 8 790 91 00
fax: + 46 8 790 90 99
e-mail: cid@nada.kth.se
URL: http://www.nada.kth.se/cid/

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## A Conceptual Framework for Computer Supported Subjective Education

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Royal Institute of Technology, 100 44 Stockholm, Sweden

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ABSTRACT: This work presents a unified pattern-based epistemological framework,called a *Knowledge Manifold*, for the description and extraction of *knowledge* from *information*. Within this framework it also presents the metaphor of the *Garden Of Knowledge* as a constructive example. Any type of *KM* is defined in terms of its *objective calibration protocols* - procedures that are implemented on top of the participating *subjective knowledge-patches*. They are the procedures of *agreement* and *obedience* that characterize the *coherence* of any type of *interaction*, and which are used here in order to formalize the concept of *participator consciousness* in terms of the *inverse-direct limit duality* of *Category Theory.*

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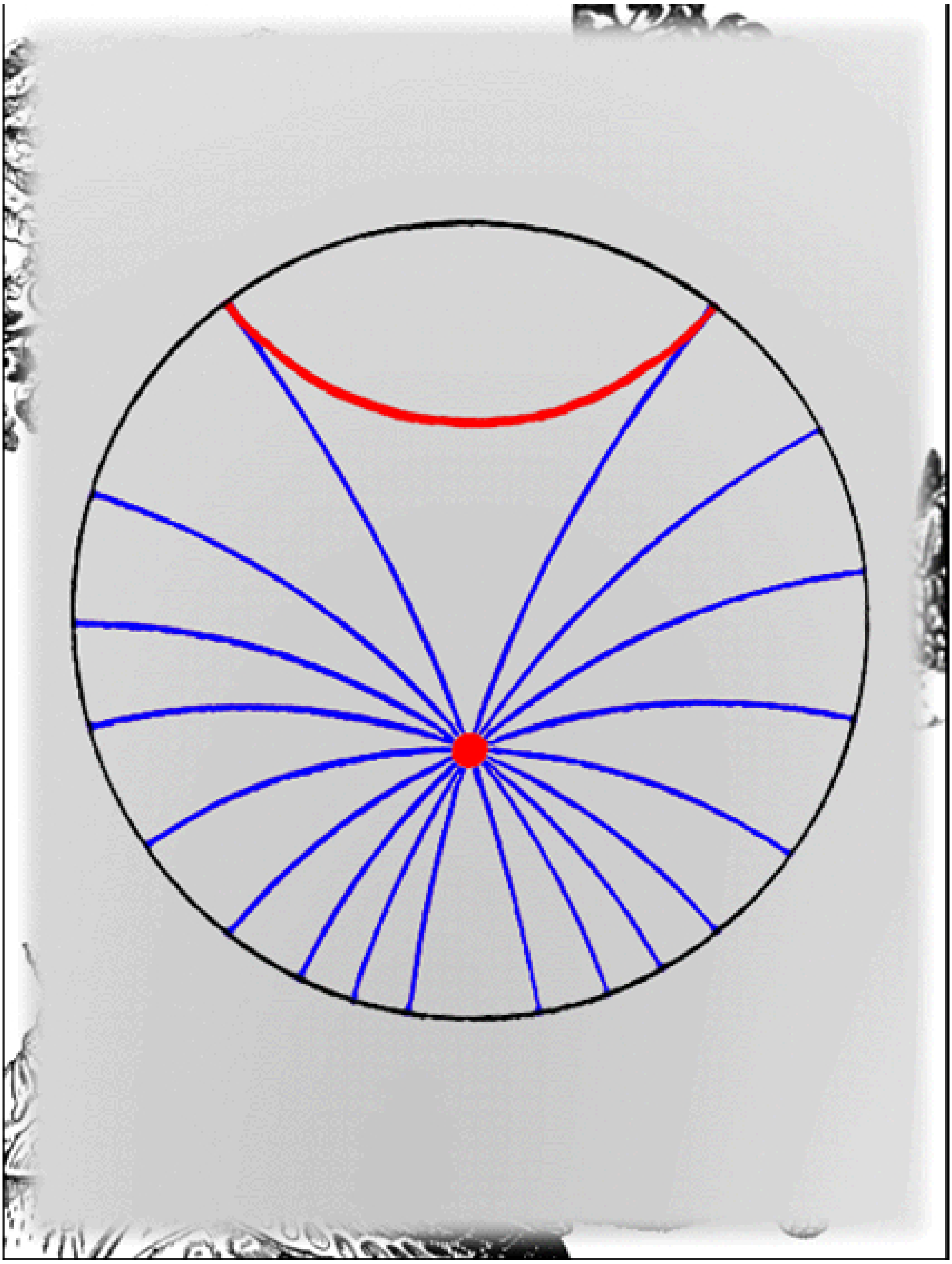

# 1 Acknowledgements

This work is the result of a constructive interaction process that has been going on over the past three decades. It has involved support and encouragement from a lot of people, to whom I am deeply indebted in many ways. I could not possibly list them all, but here are some that I can think of that represent ‘supportive influence’ along my trail.

As a mathematics teacher, I am especially grateful to two of my own teachers, *Henrik Eriksson* and *Harold Shapiro*, for their vital influence in awakening and strengthening my mathematical interests. They are by far the two best mathematics teachers that I have had the fortune to come across as a student. In 1967 *Henrik* was responsible for recruiting me into the ‘guild of mathematics teaching’ - and during the following couple of years he dumped a lot of creative opportunities in my lap, in connection with some early forms of multimedia-based mathematics teaching (the TRU adventure). Later, when I was a graduate student, *Harold* fostered my mathematical thinking and helped me bring it to new levels at a time when it was critically needed. Apart from his many deep structural insights, Harold introduced me to the historic dimensions of mathematics - by giving a course on the subject in 1973. He also deepend my philosophical interests in general - by introducing me to such profound thinkers as Clifford [(18)] and Poincaré [(128)-(131)].

As a programmer, my deepest gratitude is due to the people that taught me important things along the way - by taking the time and energy to discuss the problems of coding with me in various ways. They include *Johan Appelgren* and *Örjan Ekeberg*, who introduced me to modern workstation lisp computing, *Matti Rendahl*, who supported my first stumbling steps in the ‘painful but powerful’ art of C-programming and pointer-based arithmetic, and *Harald Winroth*, who is presently helping me sort out the ‘strange new world’ of templatified C++, as well as the ‘pros and cons’ of the *Java* versus *Tcl-Tk* approach to networked computing.

As a researcher, my special thanks are due to *Yngve Sundblad* and *Jan-Olof Eklundh*, for their unvailing support of my computing efforts, as well as my various projects in mathematical research and education in general. *Yngve* gave me my first Mac - back in 1985 - at about the same time as *Jan-Olof* gave me a position as ‘geometer’ within his Computational Vision and Active Perceptions research group (CVAP) at NADA. Over the last 12 years, this has allowed me to devote half of my working time to geometrical research. About ten years ago, *Yngve* invited me into his GUI-based programming projects - called PIM, PMI, IPM, etc - where I had the opportunity to design educational tools for mathematics, and get them implemented by a group of dedicated, creative and often surprisingly competent students. Over the last decade, these projects have resulted in quite a few ‘interesting attempts’ in computer supported mathematics education. Six years ago, *Yngve* also made a few of the old generation of Macs available to the kids of the ‘First Class Mathematics’ project which formed the basis for its vitally important computer support.

As the architect behind the Garden Of Knowledge project, I am grateful to the entire GOK-group, who have put up with my never ending flow of ideas and suggestions, and worked hard in order to separate the sense from the nonsense. The core of the group consists of *Rikard Linde*, *Kenneth Olausson*, *Katarina Skantz, Bo Westerlund* and *Kristina Åsvärn*. - with an addition of *Mattias Algotsson*, *Klara Desser-Hansson, Anders Jäderberg, Måns Tegnér* and *Fre-*

*drik Winberg* during various parts of the development process. During the spring of 97, Fredrik has done an excellent job of implementing the third prototype of the GOK in Director.

As a program-user, I am grateful to NADA's excellent Systems Group for their infallible support and dedication in solving all kinds of computational environment problems. *Ragge Sundblad*, *Jan-Erik Mångs*, *Mikael Magnusson*, *Stefan Prytz*, *Lasse Carlestam* and *Johan Berglund*, have all been especially supportive to me over the years - during the last year to the great benefit of the GOK-project as well.

The GOK-project is carried out within a stimulating environment of mixed competences, which is called the *Center for User-Oriented IT Design* (CID) at NADA. I am grateful to Yngve for the crucial part he has played in creating CID as well as for his ongoing efforts in order to maintain and develop it further. Working - since a year - at CID as a guest researcher has given me access to a creative arena where my educational ideas have found an interdisciplinary meeting-ground for constructive development. *Viveca Bjuhr* and *Sören Lenman* have been especially valuable in keeping the CID environment operational to our project in various ways. I am also grateful to Sören for illuminating discussions on various research problems concerning navigation in large information spaces [(96)], as well as on the VRML/Java based interaction possibilities of the emerging virtual classroom. This gratitude also includes *Richard Wessblad*, my friend and fellow computer consultant, for his contributions to the structure of the virtual classroom - especially in connection with its possible implementation in Java.

My heartfelt thanks also to *Lars Svensson*, my old friend and 'mathematical soulmate' from the department of Mathematics at KTH. Many ideas presented here have originated in our ongoing philosophical and educational discussion - which we have kept going since it got started back in 1975. These ideas include the mathematical definition of 'actor/reactor consciousness', in terms of the so called 'inverse' and 'direct' limits of category theory, that we formulated together during an abstract algebraic period of the early eighties[1].

Speaking of old friends, I am also indebted to *Lloyd G. Cross* and his son *Lloyd T. Cross* for our long-term collaboration on various projects - starting with solar energy and the *double-cylindrical pointfocus principle*[2] in 1976, and leading up to our present projects on holographic programming (HoloTrace) and 3d-image viewer technology (3D-screen). The discovery of the double-cylindrical mathematics back in 1976 gave me the drive to start exploring geometry on my own - as a form of geometric archeologist. This activity has been taking up a significant part of my time ever since.

I am grateful to *Anette Philipson* and *Kristina Lindgren* at *St.Erik's Catholic School* for letting me have such great fun with the kids in my daughter's class - by doing *First Class Mathematics* with them about once a week over the past six years. This experience has strengthened my conviction that mathematics can be made available to children in a new and more exiting way. Thanks also to *Hans Nihlén* of Ericsson (EUAB) who contributed to the FCM-project by lending us an LCD-display for overhead-projection of the computer screen. This made it possible for us to display and discuss the childrens individually created MacWallpaper patterns - as well as to interact with the powerful *Mathematica*™ program - in front of the class.

1. See [(106)].
2. For a mathematical formulation as well as a proof, see [(111)], pp. 154-156.

Thanks to *Hans Markstedt* at the department of Photography at KTH , who managed to produce exactly the information I wanted, namely a high-speed film recording of the splash of a drop of milk. We use it as a major multimedial attraction in the present (=third) prototype of the GOK. Thanks also to *Göran Adolfson* at Sveriges Telvision, who supplied me with some multimedial rawmaterial (= videotape) for the second GOK-prototype of December 96.

I also want to thank *Tomas Andersson*, for an interesting discussion that we had in May 96. A month later, his concept of the 'Garden of Sweden' contributed to making me realize the 'paradise-dual' kind of symmetry between the 'garden of knowledge' and the 'tree of ignorance' - on the one hand - and the traditional 'garden of ignorance' with its famous 'tree of knowledge' on the other.

Speaking of discussions, I gratefully acknowledge the 'philosophical coffee-table-club' at CVAP. Quite a few of the workhours 'wasted on philosophy' have been spent around that table - in conversations with a multitude of interesting people - including *Magnus Andersson*, *Fredrik Bergholm*, *Demetre Betsis*, *Lars Bretzner*, *Henrik Christensen*, *Jan-Olof Eklundh*, *Birgit Ekberg-Eriksson*, *Daniel Fagerström*, *Per Fornland*, *Tony Lindeberg*, *Peter Nordlund*, *Kourosh Pahlavan*, *Matti Rendahl* and *Richard Wessblad*.

Last but not least I want to thank the 'manyfold' of students that, over the years, have endured, allowed and even encouraged my various forms of conceptual, historical and philosophical deviations from the hard-core computational subject-matter of the required mathematics-curriculae. Thank you for your patience, your interest and most of all for your *enjoyment* in letting me share so many of my personal 'mathematical enthusiasms' with you!

*Man lär så länge man har elever!*

KTH, August 31, 1997

Ambjörn Naeve

# 2 Subjective Background

Since I will regard all knowledge as built by calibration on top of a subjective base [Definition (21)], it is appropriate for me to give a brief description of the subjective background that has brought me into writing these pages.

I started teaching mathematics at the university level (KTH) back in 1967. At that time I was just a teenager and a student myself, in fact starting out for my second year at the division of Technical Physics at KTH. I was hired as a so called "exercise-assistant" (övningsassistent) in a basic course of "linear algebra and geometry" - a kind of teacher that would solve problems on the blackbord in session with a group of about 15 - 30 students.

I instantly loved teaching because I got a good response from my pupils - and I consider myself extremely fortunate to have been allowed to spend all but one of my student years at KTH teaching there at the same time. By some 'simple twists of fate', already in my second teaching year - the infamous 1968 - I found myself involved with experimental (Video-based) mathematics education, and the third year (1969) I was, de facto, responsible for this new type of TV-based math-course, called the TV-course fo Linear Algebra. I ran the show on my own, so to say. We had no human lecturer and I was managing the course - as a course-assistant (of course!) - but I was in fact doing the equivalent job of a present 'högskolelektor' - like hiring and briefing exercise-assistants and putting together the course exam. The lectures were taped on huge reels and I had to go down to the control room and mount them at the start of each class. In those days we had TV-monitors all over the classrooms in 'Sing-Sing' (as the major mathematics-faculty building at KTH is called).

During the seventies, I spent a couple of years traveling around - mainly in Africa, India and North America. In 1976, I met Lloyd Cross - a physicist working in San Francisco and well renowned within the fields of lasers and holography. Working together we made a geometric discovery that changed my life. It was a new and simple way to concentrate radiative energy from a distant source by replacing the familiar parabolic disc by two parabolic cylinders.[1] This *double-cylindrical pointfocus principle* made me embark on a serious study of classical geometry. I discovered a 'lost continent' of more or less forgotten results, which had the effect of turning me into a form of 'geometric archeologist'. Some of my early rediscoveries were in the field of so called *projective geometry*, and in the early eighties I became a 'projective prophet', preaching the beauty of this branch of geometry - as well as its usefulness for the representation of many visual engineering problems. In 1983 I met Jan-Olof Eklundh, who became my first 'projective disciple' and gave me a research position within his newly formed computer vision group CVAP at KTH. The projective geometry course that I gave at CVAP in 1983 - as well as some subsequent research work[2] - marks the beginning of an evolutionary process that has lead to a widespread use of projective geometry within the field of computer vision.

Around that time I became interested in computers as a means to interactively explore and experience geometric structure [(110)]. I started out in Lisp - on a Xerox-1108 Interlisp workstation - but I also did some work on the early Macintosh[3]. When the Mac-II came out in 1987,

1. For a mathematical proof of the double-cylindrical pointfocus principle, see [(105)] or [(111)].
2. especially [(107)] and [(109)].
3. due to the courtesy of Yngve Sundblad, who cut across the red tape and supplied me with a machine for home use.

I switched to this platform, and the same year I traded the Xerox-machine for a Symbolics 3600 running Commonlisp. By that time I had started up a research project called *Motion Problems in Computer Vision*, together with Lars Svensson - a friend and fellow mathematician at KTH. This lead us into new forms of geometric archeology - resulting in work on *projective correspondence* [(161)], *exterior algebra* [(108)] and *double algebra* [(159)] - which eventually found its way into the computer vision community. However, by that time Lars and I had discovered *geometric algebra*, which seems to provide the ultimate algebraic description of geometry [(71)].

In parallel to this development I also worked a lot with something called *congruence geometry*, which is concerned with the representation of various sets of lines in space. In 1993 this became the subject of my PhD-thesis in geometry, which has the title *Focal Shape Geometry of Surfaces in Euclidean Space* [(111)]. Several of the experiments in my thesis were carried out by the use of *Reflections*, which is a program for the interactive study of reflected wavefronts, that was developed on the Symbolics machine by Johan Appelgren - as a part of his masters of engineering thesis under my supervision [(5)].

Moreover, since 1986, I have been involved with developing software for mathematics education [See chapter (8.2) ]. Also, since 1991, I have been conducting an experimental project in early mathematics education - called the *First Class Mathematics* project - at St. Erik's Catholic School in Stockholm [see chapter (6.5)].

Since 1990, I have also worked part time as a consultant for industry, solving problems in applied mathematics and implementing the solutions in C, C++ and Mathematica®. I have also developed courses in Lisp, C++ and Object-Oriented Analysis and Design - courses for industrial programmers and software engineers, which I have taught in various companies, such as e.g. Ericsson and Telia.

To summarize my teaching activities at KTH, over the last 30 years I have taught just about every type of course there is within the mathematics faculty - *algebra*, *real analysis* in one and several dimensions, *complex analysis*, *fourier analysis* and *discrete mathematics* to name a few. During this time, I have been engaged in the inside 'back-bone' work of mathematics teaching, such as the manufacturing and correcting of exams and the negotiations involved in establishing their 'calibration protocols' of error-tolerance. It is this involvement that forms the basis from which I speak on the subject of education in general, and of mathematics-education in particular.

# 3 Introduction

## 3.1 Purpose and Axioms

This paper is about spreading an infection. A kind of conceptual ‘idea-virus’ that has to do with how to think about the educational process. The infection manifests itself in a change of ‘thought-pattern’ - away from the traditional *compulsory* learning pattern with its *examination*-oriented quality metri [Figure (8)], towards a new pattern based on *voluntary* learning with what I call *insamination*-oriented quality metric [Figure (9)]. This is a quality metric based on *‘interest and entrance-tests’* - instead of the traditional *‘grades and exit-tests’* that form the backbone of the old quality metric.

In Chapter (11) below, I introduce the concept of a *Knowledge Manifold* (KM), as an abstract framework within which to think about knowledge formation in general. The fundamentally important property about a KM is that it builds knowledge *subjectively*, i.e. from the individual (*subjective*) towards the collective (*objective*). Such ‘calibration processes’ are performed by establishing and adhering to a variety of extremely complex ‘calibration-protocols’.

Moreover, I discuss a specific model (= realization) of a KM called the *Garden Of Knowledge* (GOK), which started last spring as a Pythagorean lecture on the historic relationships between mathematics and music - and has evolved into a computer based interactive learning environment that is presently being developed at CID[1]. The GOK-project has given me an opportunity to subject my educational ideas to many forms of ‘constructive filtering’ by an interdisciplinary group of creative people in a highly stimulating environment. This has contributed substantially towards transforming these ideas into communicable form.

Speaking from three decades of teaching and research experience within the fields of mathematics and computer science at KTH, having seen a generation of ‘teknologer’ pass by in their ‘perennial hunt for degrees and diplomas’, I have taken the liberty to present here some of my personal views on the subject of education. These views are presented within a framework of conceptual structure, which I believe to be important in connection with any serious discussion of the structure of the educational process as a whole. They are also important as a conceptual foundation from which to think about the ‘electronic evolutionary possibilities’ of traditional teaching which are manifesting themselves in the emerging ‘virtual classroom’.

Over the years, I have seen the emphasis of the mathematics courses shift. Basically, the shift has taken place in one direction: towards *How* and away from *Why.* In terms of educational mathematical activities, this has resulted in more and more of *algorithms* and and less and less of *proof.* No doubt, the emergence of the computer has contributed substantially to this development, which may well be one of the reasons why it holds such low esteem in mathematical quarters. Sharing - to a degree - this ‘mathematician’s contempt’ for the computer, I have at the same time become convinced that the computer has great potential as a tool to substantially increase the level of conceptual (= mathematical) understanding. Hence, during the last decade, I have been involved in designing computer-based tools that support mathematics education in various ways.

1. See the group-report [(97)] from the Garden Of Knowledge project, KTH, September 1997.

Working, since my daughter's first school year, with her class as an 'external resource' in mathematics has given me a unique opportunity to test some of my mathematics-educational ideas on a few young and un-biased minds. This experience has strengthend my conviction that one can introduce essential mathematical ideas from the start - first class mathematical ideas from first class on, so to say - and do so in an interest-provoking way. Over the last six years, I have presented these kids with a kind of 'mathematical smorgasbord' of structural ideas. This is my small-scale version of the *First Class Mathematics* project, and it forms another experimental basis from which I approach the subject of mathematics education.

On one level, this paper contains a number of subjective statements, which is why I have chosen to write it in first tense. However, this does not imply that the paper is devoid of 'objective content'. On the meta-level, I have tried to separate these two qualities by introducing the labels **Opinion**, for the *subjective* and **Fact** for the *objective* aspects of knowledge - as I conceive them. Of course, there are no sharp limits, only a gradual difference in types of calibration procedure.

My discussion starts out from *two fundamental opinions* that I have elevated to the status of **Axiom** (= *fundamental belief* ). They represent the ultimate foundation upon which my thinking about education is built. If you do not share my belief that these axioms are important guidelines for the educational process as a whole, then you will probably disagree with most of what I have to say further on. On the other hand, if you do share this belief, chances are that my ways of reasoning will have more in common with your own.

**Axiom 1:** *No one but you yourself can teach you anything.*
*A good teacher can inspire you to learn.*

We all know that the most effective way for a teacher to get results is by getting the learners interested. And this, in turn, is most effectively achieved if the teacher can manage to display her own love of the subject.

**Axiom 2:** *It should be anybody's right to refuse to be taught,*
*without thereby losing respect from society in any way.*

No none should be taught anything against his or her own will. The right to refuse to be taught is an important human right, which is seriously endangered these days. The information-society is being force fed down our collective throat, whether we want it or not. That is why IT is invariably portrayed as a form of 'human right' for everybody - in order to mask the underlying issues and avoid the embarrassing debate of whether we really want this IT-based society or not. And - in case we do want IT - what do we really want IT for? Or, in other terms:

**Question 1:** What are we really going to inform each other about in this information age?

Let me venture my first formal

**Opinion 1:** The lack of debate around the vitally important **Question (1)** reflects the predominating IT investment pattern, which can be expressed as the following variation on Marshall McLuhan[1]: *We've got the media - but where is the message?*

In our days, *information* has become identified with *negative complication* - in the sense of *removal of uncertainty*. The *informational content* of a certain message - its so called *Shannon-information* - is measured[1] in terms of how long it takes to transfer the message on a given communications line. Tor Nørretranders has noted the interesting fact that the two people who created the dominating information concept of the electronic age - *Claude Shannon* and *Warren Weaver* - both worked for AT&T. This company is of course a giant among the telecom companies that are now reaping the benefits of the present *information => complication* process - as they busily transform McLuhan's vision of the *global electronic village*[2] into their own vision of the *global telephone kiosk*.

**Opinion 2:** The global-telephone-kiosk pattern shows strong similarities to the pattern of the familiar 'computer-in-the-classroom' neurosis, which in turn is similar to the dominating thought-pattern of traditional development aid. Just as Africa has beed filled with rusting tractors - rendered disfunctional by lack of infrastructure, concentrating on sales-talk of type "each kid should have a personal computer" has the result of drowning the class-rooms with 'rusting computers' of all sorts - rendered disfunctional by lack of teacher training. Too often the computers end up just standing there as a form of teacher's alibi for not having to deal with this weird and scary computer thing anyway.

Moreover, while I'm at it, let me recall

**Opinion 3:** The computer has great potential as an educational tool - in support of creative exploration - not only within the field of mathematics, but in many other fields as well.

However, I have no doubt that

**Opinion 4:** If the educational potential of the computer is to be put to effective use within the school system, it implies at least a doubling of the 'live-teacher-density', i.e. a halving of the student group size.

When the students start exploring on their own, the sheer number of their questions can be expected to grow dramatically. After all, this is one of the major advantages of an individually oriented curriculum - in as much as it indicates some form of ongoing mental activity! If the school-system doesn´t provide live resources to respond to this wave of questions, computer-based education is bound to increase the level of confusion and frustration in the minds of the students - instead of promoting their interest and curiosity to explore further!

1. '*'The media is the message'*, see [(102)].
1. For a mathematical definition of Shannon-information, see [(149)].
2. See [(103)].

## 3.2 An Overview of The Garden Of Knowledge

### 3.2.1 What is the Garden of Knowledge?

**Definition 1:** The *Garden Of Knowledge* in the following referred to as the GOK - is a computer program for keeping track of the interrelated structure of ideas, designed to support the expression of their relations to other ideas as well as their evolution over time and culture.

The GOK can be seen as a multi-medial tool which aims to develop an interdisciplinary understanding of the world of phenomena by supporting their *conceptualization*, *exploration* and *explanation* in an *experimentally* oriented way. Its aim is to help its users to “Ask every question and question every answer” within the chosen field of his or her own interest.

The purpose of the GOK is to expose and illuminate the concepts which forms the basis of the science-oriented worldview which characterizes our modern industrial society.The GOK aims to work as a sort of *philosophy s(t)imulator*, contributing towards providing a net-based structure for the ongoing philosophical debate which is collectively knows as the educational process. The GOK can therefore be regarded as a knowledge-tool which is especially well suited for an interactive and individual-oriented distance education on the net.

The GOK purports to aid the user to *ask every question* and *question every answer*, within the chosen field of his or her own interest. Hence it supports an educational process wich aims to reduce the final authority of the answer by *responding* to (instead of answering) a question - and thereby assuming more responsibility for it.

The version of the GOK which is presently being implemented is specifically devoted to illuminating the connections between *mathematics* and *music* - regarded both from a *conceptual* and from a *historical* perspective. This brings us back to the 6:th century BC and to ‘the old Greeks’. The mathemagician *Pythagoras* is therefore a central figure in the GOK, both as the discoverer of the mathematical foundations of *musical harmony*, as well as the instigator of a contemplatively oriented, science-based religion, which regarded the contemplation of the *rational mystic of numbers* as a way to attain the deepest form of knowledge.

### 3.2.2 What does the Garden Of Knowledge consist of?

The GOK consists of a mixture of *theoretical* and *experimental* environments. The theoretical parts in themselves consist of entries concerning *phenomena*, *concepts* and *people*. Working in the Garden Of Knowledge is called *studying*, and a GOK-worker is referred to as a *learner* or a *student*. Studying includes recording ones responses to different types of questions that relate to a potentially unlimited number of *phenomena* and *concepts*. To encourage the questioning process, a few default question/response configurations are available to the student.

The overall question is phrased as: “What is its nature?”, which is magnified by a What--How--Why aspect-coordinate-system with the respective dimensions: “What does it consist of?”, “How does it behave?”, and “Why does it behave this way?”

### 3.2.3 How does the Garden Of Knowledge Operate?

The teacher-gardener maintains her own garden of knowledge, where she alone has root priviledges. Her thoughts on various questions are always available to the guests of her garden in the form of “thoughts from the root”.

Students are invited into the teacher´s garden and given the opportunity to respond to various types of questions, some of them “defaulted” by the teacher-gardener, others constructed by the students themselves. The questions fall into different basic categories: The questions of WHAT (*analysis*), HOW (*design*), WHEN (*strategy*), WHY (*ethics*), WHICH (*esthetics*), WHERE (*trend*), WHO (*acknowledgement*) and IF (*hypothesis*) respectively. The students are also free to comment on the teacher’s root-responses to these questions. When the student is engaged in this activity, any pre-recorded root-responses of the teacher-gardner is always available, making it possible to follow their explanatory links further into the literature.

The GOK has three different modes of operation - corresponding to three different kinds of trees (apples) - namely the *exoteric* mode (corresponding to the tree of *materia*), the *esoteric* mode (corresponding to the tree of *spirit*) and the *random* mode (corresponding to the tree of *ignorance*).

Each student must “take a bite” from one of the three apples before any questions can be asked. This act transforms the garden into the corresponding mode of operation, which has an influence on the formulation of the question itself. Thus when the overall question is asked in the *exoteric* mode, it is formulated as: “What is its *material* nature?”, whereas when the same question is asked in the *esoteric* mode, it is formulated as: “What is its *spiritual* nature?”. In the *random* mode, both the questions and the responses are created as syntactically correct verbal sentences (well formed formulas) with their individual members drawn at random from a database of candidates from different word classes[1].

### 3.2.4 What is the Purpose of the Garden Of Knowledge?

The purpose of the GOK is to cultivate knowledge and make it transmute into understanding. This means supporting the transformation of the traditional teacher-preacher into a new type of teacher-gardner (teacher-guardian) of knowledge. Just as the catepillar has to retire into a cocoon in order to transmute into a butterfly, so must the student engage in spinning his mental cocoons inside the (world wide) web of knowledge, retire inside it, reflect upon his thoughts and internalize them - in order to make them aquire the wings of understanding! Within his garden, the teacher-guardian is devoted to guarding these cocoons, and nourishing them well in order to support their delicate inner development process!

1. This is a contraption that could be thought of as a kind of *one-armed verbal bandit*, whose jackpots are delivered in the form of ‘silly poems’.

## 3.3 The Concept of a Knowledge Manifold

An important idea presented in this paper is the concept of *an idea as a representation of a subjective experience* [Definition (21)]. This represents a totally anti-Platonian definition of an idea. Plato's ideas were objective and eternally true' - although they were only available to us by contemplation of the mathematical mysteries. The definition of idea used here is totally subjective and represents each individual's own 'mental space'. The collection of such ideas - within each individual - is called a *knowledge-patch*.

Such subjective knowledge-patches do not grow in isolation. Although each person is the gardner of his or her own personal knowledge-patch, these patches are constantly calibrated with their surroundings in a multitude of different ways.

In Chapter (11) I introduce the term *Knowledge-Manifold* for the *collection of calibrated knowledge patches*. The concept is inspired by differential geometry, where the equivalent of a knowledge-patch is called a *local-coordinate-patch* - and represents a way to describe (= parametrize) the local neighborhood of some point. The famous mathematician David Hilbert has commented that "most people's thoughts move around in a circle with radius zero, which they call their stand-point". In the context of a Knowledge Manifold, one can expand on Hilbert's observation and say: People's thoughts move around in a space of personal ideas, which could be called their knowledge-patch or their own subjective reality.

# 4 Terminology and Notation

## 4.1 Thought Patterns

In the following pages I will make use of various types of *patterns* - a way of expressing structure that is becoming popular within the software engineering community[1]. However, it is also receiving attention in wider circles - even including such 'mathematically remote' areas as the humanistic sciences[2]. Since I will make use of patterns in order to express various combinations of conceptual thoughts, I make the following

**Definition 2:** A *thought-pattern* expresses the *interaction* of a number of *concepts*. It represents a way to think about the underlying subject matter.

A thought-pattern is a form of *arithmomorphic grid* [3]that we put on top of our *dialectical experiences* in order to make them *communicable* (= *medial*). The most obvious example is *language* itself. As a thought-pattern, our language shapes our way of thinking in more ways than we could ever express[4]. In fact, it determines the very essence of what is *expressible* ('sagbar') in the sense of Wittgenstein, who finishes his Tractatus with the famous words:

> On whatever subject we cannot speak, we must remain silent[5]

In accordance with the terminology of 'arithmomorphology', I make the following

**Definition 3:** The term *arithmomorphic distortion* will refer to the *discrepancy* that any *arithmomorphic grid* (= thought-pattern) by necessity creates in relation to the underlying *dialectical reality* that it is trying to *express.*

A thought-pattern functions as a grid by trying to capture (= express) some kind of typical behaviour. It functions as a form of 'strategic mistake'. By trying to express something but not quite succeeding, it provokes a discussion about its own inappropriateness. In this way a thought-pattern directs mental energy towards the process of *making its distortions expressible*, thereby inspiring a more informed calibration process between the different participators involved in its interpretation.

A thought-pattern is always subjectively created, and successively objectivized by different calibration procedures. The state of objectivization (= collective agreement on validity) of any though-pattern is represented by its corresponding calibration protocols. Since the thought-patterns presented here lack an attempt to describe these protocols, they are to be considered basically my own subjective thought-patterns. That does not mean, however, that I consider them all

1. See e.g. [(50)].
2. They have been used e.g. within literature criticism in order to construct a kind of 'pattern protocols' for gender based socal interaction. See [(116)] or [(117)].
3. The dichotomy between *arithmomorphic* and *dialectical* concepts is due to *Georgescu-Roegen*, [(57)], who uses the term *arithmomorphic* for sharply defined concepts, that are suitable rawmaterial for the 'either-or' type of reasoning of *Aristotelian logic* - as opposed to the *dialectical* concepts, that are smoothly transcending and gradually overlapping, like e.g.' day' and 'night', 'truth' and 'deception', or 'joy' and 'sadness'.
4. See e.g. [(170)].
5. "Wovon man nicht sprechen kann, darüber muß man schweigen", [(174)], p. 115.

as having originated with me. On the contrary, I have often tried to express how I conceive the thought-patterns of others. This is of course an important part of the objective calibration process.

To summarize this discussion, I want to emphasize the following

**Fact 1:** When I express a certain thought-pattern, I do not mean it to be taken literally. Neither do I claim it to apply to any single individual case. The inevitable arithmomorphic distortion of Definition (3) must always be taken into account.

I see though-patterns as stereotyped ways of thinking that may or may not apply in the individual case, but which might have a certain ‘statistical quality’ about them. In fact, I think of them somewhat like the laws of quantum mechanics in contrast to the old deterministic laws of classical (= newtonian) mechanics.

The common denominator of the thought-patterns that I present in this paper, is the fact that I have seen traces of them in various places - both within academia - in the small - and within the surrounding society itself - in the large. In some cases this may well have been due to my misconception of the situation. In any case, it is an important benefit of thought-patterns to contribute towards making all forms of prejudice explicit.

### 4.2 Object Modeling Technique

The notation used in this paper is standard OMT [1], with a few notable exceptions.

The *type identificaton operator* - which maps an *instance* to its corresponding *type* - is denoted by **isA** , and its inverse, the *instantiation operator* is denoted by **a**.This brings the model closer to the normal linguistic representations of the corresponding concepts, as should be evident from Figure (1). I also mix the instance- and the class-diagrams of OMT in some ‘ways of my own’, as you will see below.

As my first example, let me take the educational shift from *Why* towards *How* - and the corresponding shift in mathematical classroom activity from *Proof* toward *Algorithm* - that I spoke of before.This structure is expressed in the pattern of Figure (2) - where the arrows are additions of my own (in relation to standard OMT):

The vertical arrows represent *equivalence* (= *isomorphy*) in the traditional mathematical sense. The horizontal filled arrows represent *trend*, i.e. a movement in time, or a tendency to shift away from something and towards something else.

1. as defined e.g. in [(138)].

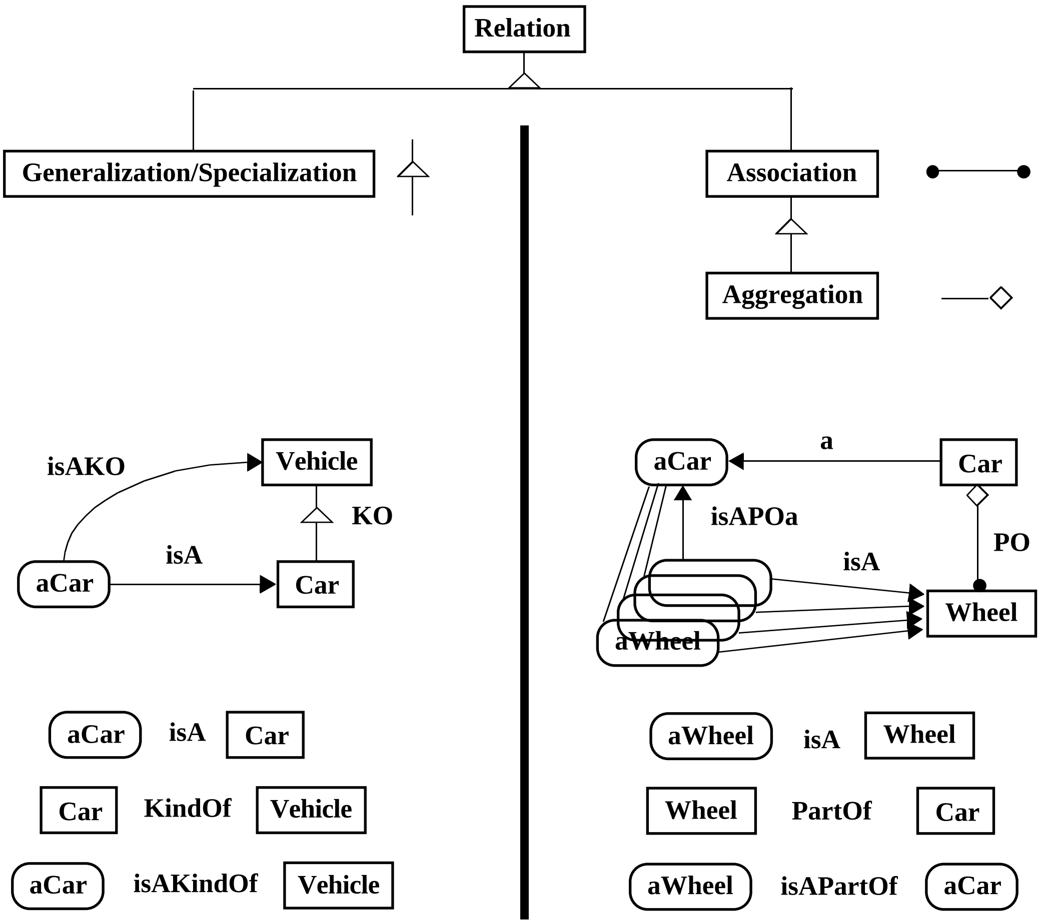


**Fig. 1. The OMT notation with the addition of isA, isAKO and isAPOa..**

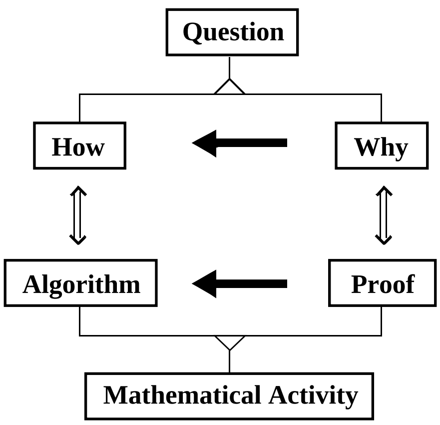


**Fig. 2. The trend in higher mathematics education as I have experienced it.**

# 5 Problem Background

The problem of maintaining the quality of the educational process has been a fundamental concern within all school-systems - ‘quality’ being taken, through the ages, to mean a variety of different things. A few hundred years ago - in the *earlier* parts of the process often referred to as *lower education* - quality was roughly equivalent to ‘knowing the 10 commandments’ as well as a couple of hymns, the more the better. Around the middle of the last century, the concept of quality in lower education also included ‘being able to read and write’ - at least to the extent of being able to follow the posted instructions at work. In the information age of today, reading and writing have expanded into necessary instruments of communication, and mastering them is therefore considered to be a vital part of each persons individual development.

In the *later* parts of the educational process - usually referred to as *higher education* - the measures of quality have been less explicit and harder to formulate. ‘Producing teachers that are able to supply the required quality of lower education’ is an obvious goal, but how should it be achieved? An what does it really mean to produce researchers of an ‘internationally competitive standard’ or some equivalently vague formulation.

An important aspect of the quality of higher education is the inherent tension between the specialist and the generalist perspective. Reading some of the classical ‘gems’ from the science-literature of about a century ago[1], one is bound to be struck by the ‘holistic ambitions’ to comprehend the world in its entirety that still existed among the leading scientific thinkers of the late nineteenth century.

But then, something amazing happened - something that fundamentally changed everything. Around the turn of the century, there was a sudden ‘explosion of abstraction’ - a kind of mental supernova - which had an enormous impact that is still being felt throughout our entire culture. To mention just a few of its many consequences, mathematics was catapulted into new conceptual dimensions, where it remained in order to explore a multitude of new and exiting structures. New fields popped up like mushrooms, resulting in such linguistic combinations as ‘point-set topology’, ‘functional analysis’ or ‘algebraic geometry’, to name but a few of the many ‘new brands’ of mathematics that were invented.

The new conceptual outlook created a strong movement towards the formalization of mathematics itself; a movement that was headed by the great mathematician David Hilbert. However, these attempts were abruptly and permanently ended by the famous article of Kurt Gödel [(62)] in 1931, where he wrote the observing subject into the mathematical equations. In doing so, Gödel gave mathematics a place among the anthropocentric activities - he so to say ‘subjectified the subject’ once and for all!

During the present century, mathematics has evolved into some kind of abstract, conceptual ‘cooking competition’, where the structural molecules of the common (= everyday) spices are taken apart and combined into new and exiting tastes. Part of what this process has done to the concepts of *continuity* and *arithmetic* is illustrated in Figure (3).

1. People like Helmholz, Maxwell and Poincaré - to name a few.

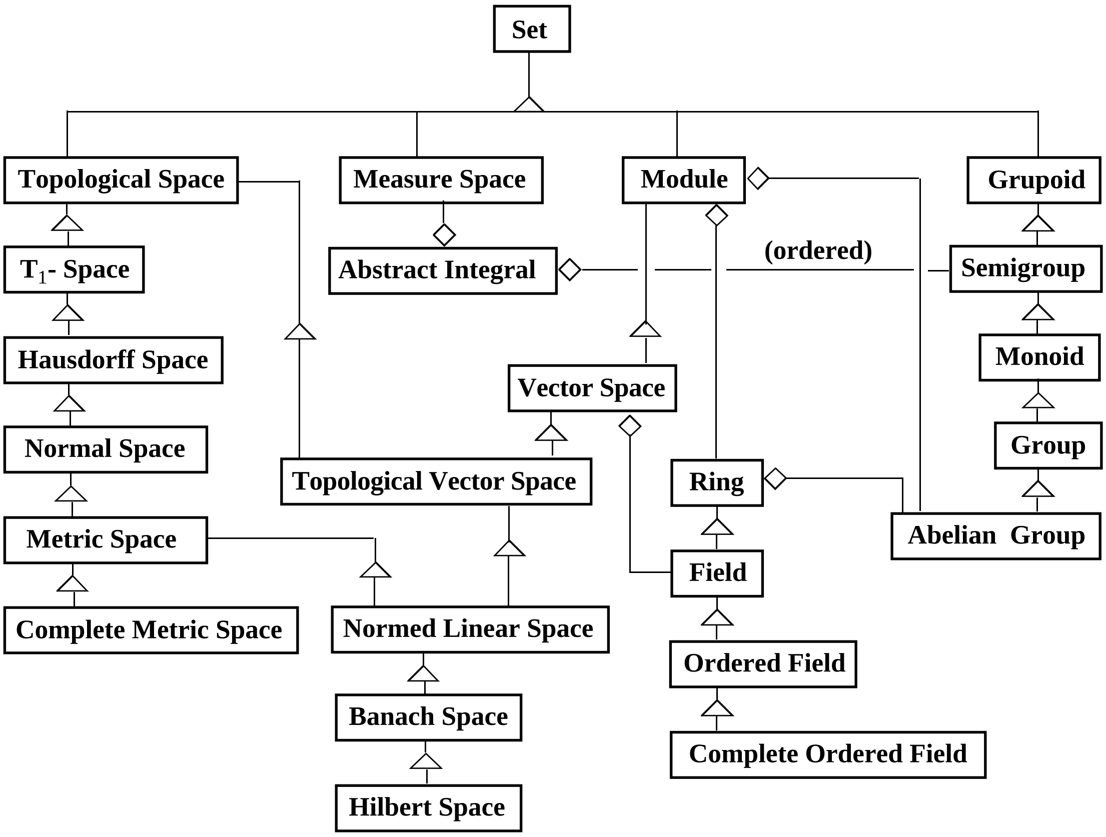


**Fig. 3. Modern mathematical taxonomy of the concepts of *continuity* and *arithmetic*.**

In physics, the effects of the abstraction-supernova included the destruction of the classical (newtonian) world-view with its God-given deterministic laws, both in the large - by *relativity theory* - and in the small - by *quantum theory* . In this mind-boggling process, the old deterministic God has been ‘randomized’ and turned into a kind of ‘hedging expert’ - who is running the world by ‘betting on averages’, and who cannot even keep separate track of space and time! “God doesn’t play dice with the universe[1]” echoes the famous words of Einstein - as a remainder of the old deterministic paradigm. But today it is ‘Order out of Chaos’, and ‘Random Rules’ - the dice games are played everywhere[2] from the sub-atomic level up to the ‘bingo-lotto’ numbered particles that bounce around on our TV-screens!

Another effect of the ‘abstract explosion’ a century ago has been to increase scientific activities by several orders of magnitude. This *‘knowledge-explosion’* is the cause of the present ‘age of specialization’. Today it is impossible for any single mind to even begin to comprehend the totality of what is going on - in order to obtain some kind of scientifically based ‘world-view’ in

1. “because he neither has the time nor the space”, one is tempted to add (from the relativistic point of view).
2. It is part of the ‘Faustian contract’ of probability theory that by giving up our ambition to aquire knowledge about the individual case, we gain a new type of knowledge that can be used to make powerful predictions on another level. Davis & Hersh refer to the social consequences of this process as the *stochastization of the world*. See [(32)], p.19. The misuse of this exceptional power is often referred to as the ‘*tyranny of averages*’.

the sense of the thinkers of a hundred years ago. Instead, we have to content ourselves with much more humble ambitions in our understanding of the human condition.

Unfortunately, this age of specialization has fostered an attitude where the attempts of interdisciplinary understanding have been largely abandoned - giving way to the opposite attitude, the well-known way of the *specialist*. In one of his philosofical essays[1] *Science and Humanism*, Erwin Schrödinger discusses, among other things, the problems of specialization. He refers the reader to an article of the Spanish philosopher José Ortega y Gasset, called *La barbarie del 'especialismo'*, where he paints the picture of the specialized scientist as the typical representative of the brute ignorant rabble - the hombre masa (mass-man) - who endangers the survival of true civilization. In the translation of Schrödinger [(145), p.110], Ortega writes

> He is a person who, of all the things that a truly educated person ought to know of, is familiar only with one particular science, nay even of this science only that small portion is known to him in which he himself is engaged in research. He reaches the point where he proclaims it a virtue not to take any notice of all that remains outside the narrow domain he himself cultivates, and denounces as *dilettantist* the curiosity that aims at the synthesis of all knowledge.
>
> It comes to pass that he, secluded in the narrowness of his field of vision, actually succeeds in discovering new facts and in promoting his science (which he hardly knows) and promoting along with it the integrated human thought - which he with full determination ignores. How has anything like this been possible, and how does it continue to be possible? For we must strongly underline the inordinateness of this undeniable fact: experimental science has been advanced to a considerable extent by the work of fabulously mediocre and even less than mediocre persons.

Leaving Ortega, Shrödinger continues:

> I shall not continue the quotation, but I strongly recommend you to get hold of the book and continue for yourself. In the twenty-odd years that have passed since its first publication, I have noticed very promising traces of opposition to the deplorable state of affairs denounced by Ortega. Not that we can avoid specialization altogether; that is impossible if we want to get on. Yet the awareness that specialization is not a virtue but an unavoidable evil is gaining ground, the awareness that all specialized research has real value only in the context of the integrated totality of knowledge. The voices become fainter and fainter that accuse a man of dilettantism who dares to think and speak and write on topics that require more than the special training for which he is 'licenced' or 'qualified'. And any loud barking at such attempts comes from very special quarters of two types - either very scientific or very unscientific quarters - and the reasons for the barking are in both cases translucent.

A little later, Schrödinger emphazises that each lecturer should possess:

> The ability to see the limits of his subject matter. In his teaching to make the students aware of these limits, and to show them that beyond these limits forces come into play which are no longer entirely rational, but arise out of life and human society itself.

Schrödinger closes his discussion of the specialist-generalist dilemma with the following words: [(145), p. 112]:

> Never lose sight of the role that your particular subject has within the great performance of the tragi-comedy of human life; keep in touch with life - not so much with practical life as with the ideal background of life, which is ever so much more important; and, *Keep life in touch with you*. If you cannot - in the long run - tell everyone what you have been doing, your doing has been worthless.

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1. Schrödinger, *Science and Humanism*, Lectures at the Dublin Institute for Advanced Studies, 1950. [(145)].

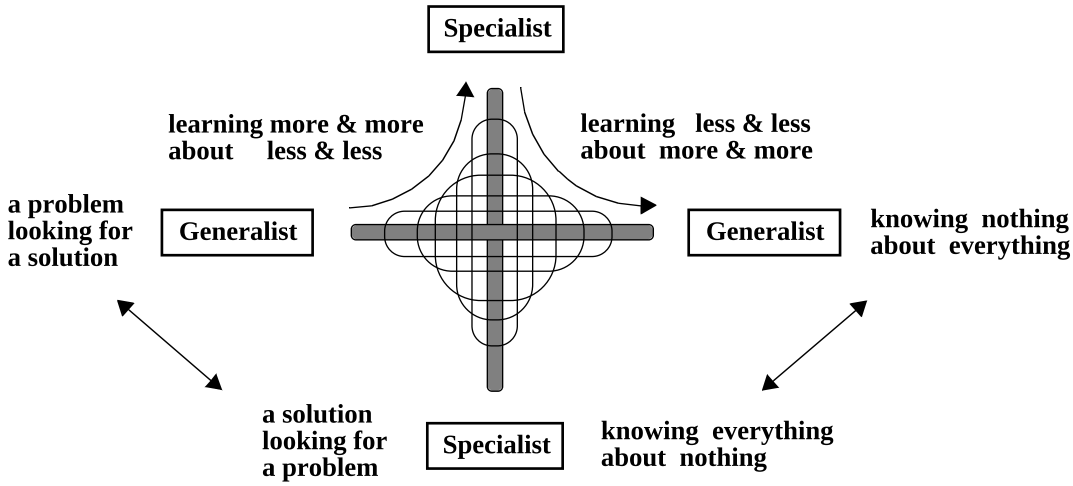


**Fig. 4. The Specialist / Generalist Duality.**

<u>Interpretation of Fig. (4):</u> This is the dilemma facing every single individual: Am I going to 'dig deeper and deeper' - *learning more and more about less and less*, or am I going to 'look wider and wider' - *learning less and less about more and more*. In the first case, I end up as a *specialist*: *knowing everything about nothing* - and in the second as a *generalist*: *knowing nothing about everything*. As a specialist, I have become *a solution looking for a problem* (= where can I apply my knowledge), while as a generalist, I have become *a problem looking for a solution* (= I see what's wrong, but not what can be done about it)..

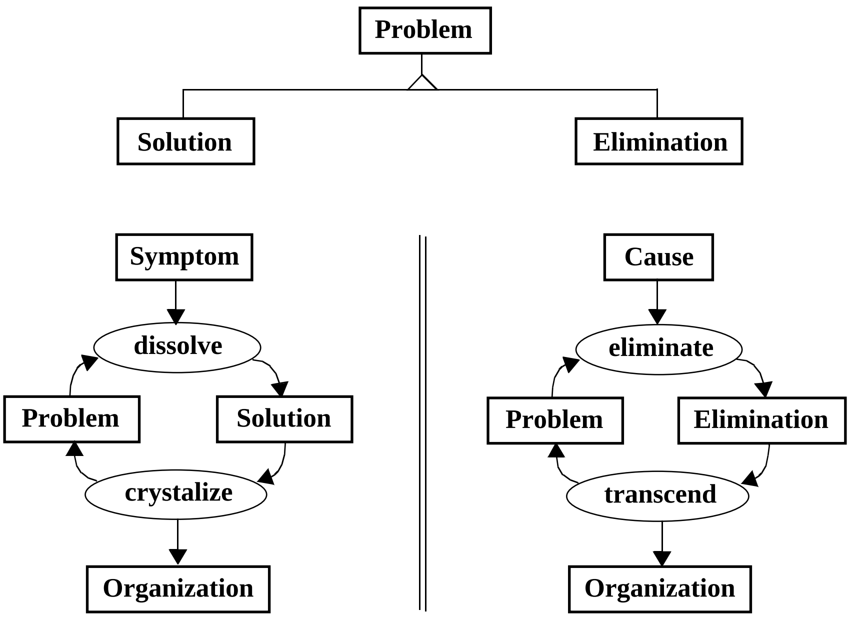


**Fig. 5. The *Problem/Solution* versus the *Problem/Elimination* pattern.**

Interpretation of Figure (5): The pattern is concerned with the difference between *solving* and *eliminating* problems. *Solving problems* has the effect of making their *symptoms dissolve*, which means to 'disappear in solution' - just like salt-crystals in water. However, from this *solution* the *symptoms* of the *problem* can be easily *crystallized* in various ways. This tends to *crystallize* (= institutionalize) any *organization* that has been formed in order to *solve problems. Eliminating problems* means dealing with *causes* instead of symptoms - which tends to *transcend* the *problems* as well as the *organization* . In fact, eliminating the problems *dissolves the organization* - instead of crystallizing it. Instead of becoming institutionalized, the organization 'dissolves into solution' - from which it can be conveniently re-crystallized if the problem should ever show up again.

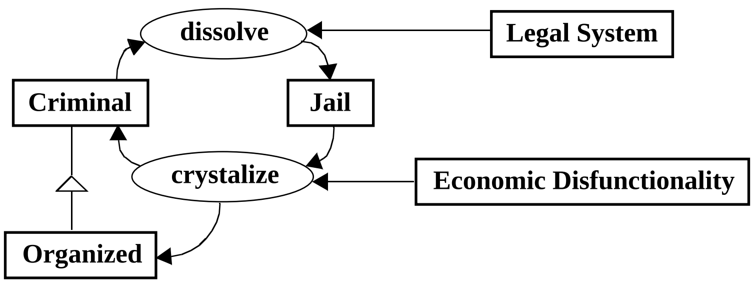


**Fig. 6. Part of the Problem/Solution pattern applied to the Judicial System.**

Interpretation of Figure (6): The judicial system is seen as a way to *solve* the problems of crime by confining the *criminals* in *jail*. This activity is supported by the *legal system*, and the *re-crystallization* of convicted criminals is assured by various forms of *economic disfunctionality*, most of all the natural reluctance to employ an ex-convict. Many criminals live their lives within this loop (= 'volta'), while others *organize* and set up some kind of legal front in order to break the cycle. In this way the *organized criminal* is seen as a form of evolutionary outgrowth - supported by the problem/solution pattern.

We are all familiar with the highly complex and intricate levels of technology that surround us in our every day lives. In fact, we acknowledge this awareness whenever we say that we live in a *'high-tech' society*. However, we also live in a time when such cold and nerdy things as mathematics, physics and chemistry do not seem to attract enough interest among the young to be able to compete with more hot and trendy things - often concerned with how to express oneself creatively by using various forms of electronic media.[1]

Hence, in our days, the problem of maintaining the quality of the educational process has become closely linked to the problem of *'raising and maintaining interest in science and technology'*. Obviously our high-tech-society is in desperate need of its share of 'dedicated nerds'. It needs them in order to perform functions that are vital for our very survival. To put it bluntly, we simply need to 'keep our bridges up, our planes in the air and our pressure-tanks sealed tight', to name but a few of our non-negotiable needs. We must cater to these needs in order to

1. This phenomenon seems to be related to what the trend-guru Malcolm MacLaren refers to as 'the ultimate Karaoke' [see Dagens Nyheter, March 28, 1997].

maintain our high-tech material lifestyle - the basic *tonal* of our technology. And this doesn't just happen by itself; it requires an army of willing and able engineers that are dedicated to the quality of their competence and profession, which means loving that cold and nerdy science-stuff. Within the entire western culture, there are consistent indications that this army-of-engineers is finding it progressively harder to enlist suitable recruits. These have to be found among young people who are willing and able to dedicate themselves to developing their mental comprehension concerning what is going on in the 'real world', and not just devoting themselves to 'living out their feelings' by 'being creative' in one way or another.

These difficulties are related to one of the classical patterns of academia - the Despise&Stigmatize pattern of Figure (7), which underlies the familiar academic tension between science and humaniora. It has its roots in the division between *humanists* (= disciples of humaniora) and *scientists* (= disciples of science). A division which implicitly indicates that a scientist is a kind of non-human(ist), indicates a very unfortunate choice of terminology. In my mind

**Opinion 5:** The dichotomy of *humanist* versus *scientist* is contributing towards the strengthening of non-human forces of technology - making man adapt to the machine instead of vice versa[1].

An attempt to capture some aspects of the 'traditional animosity' between science and humaniora is presented in Figure (7).

<u>Interpretation of Figure (7):</u> Within *academia*, the subdiciplines called *science* and *humaniora* are viewed as interlocked in a destructive pattern of *contempt* and *demonization*: The devotees of *science despise* the disciples of *humaniora*, and the latter reply by *stigmatizing* the former, and depriving them of their social, emotional and sexual value as human beings. This pattern creates the concept of the *nerd* as an encapsulation of *un-sexy-ness*.

The rock-part of the pattern is a 'free-association' on top of the pattern - representing a play of words that has a certain relationship to the subject matter of the pattern. *Einstein* and *Frankenstein* are mentionend here as rolemodels for the 'un-understandable,' respectively the 'irresistable urge to find out' - both of which are qualities that are presently associated with the nerd.
As personal characters, however, both of them were rather atypical nerds, and are reknowned to have asserted quite an attraction on the opposite sex. Of course, the actors playing Frankenstein have all been quite sexy - a quality that has been attributed even to his monster!

So the rock-part of the pattern breaks down at this point, which illustrates how hard it is to apply any thought-pattern to people.

1. In fact, bringing humanists and esteticists into the IT creation process is a major purpose of CID, thereby strengthening the humanistic influence over the design of information technology in general.

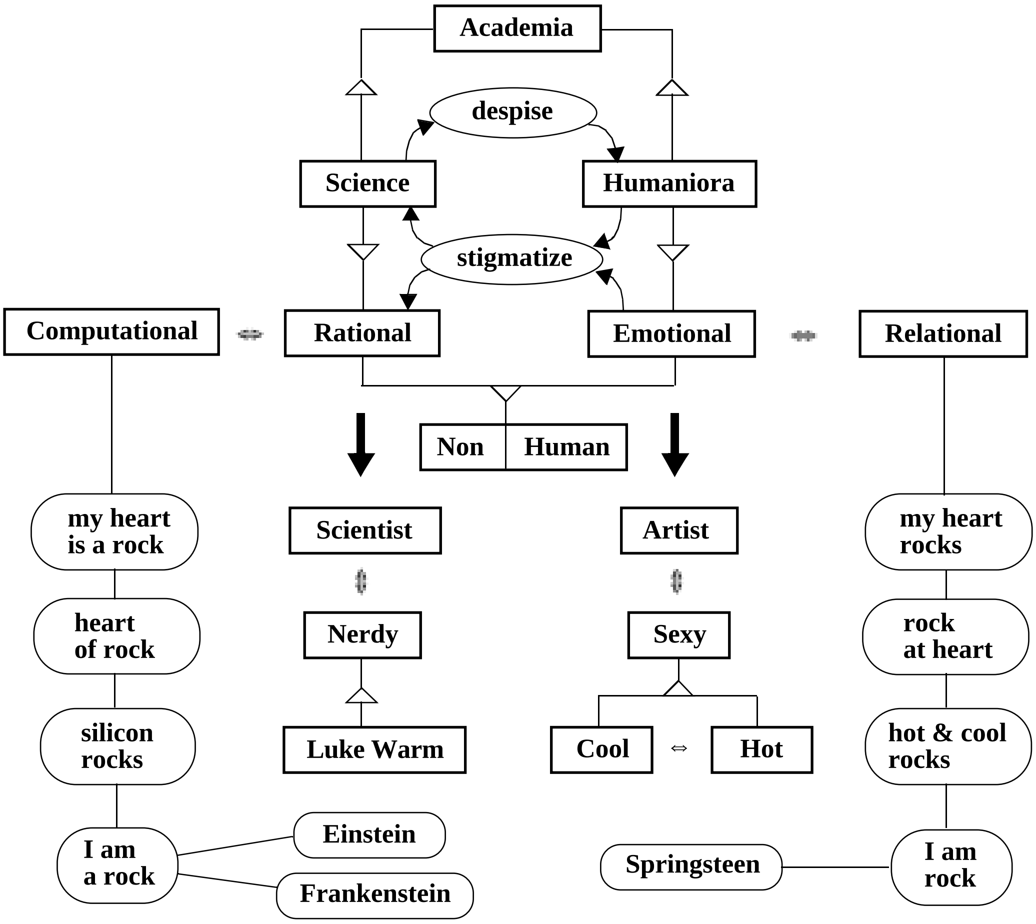


**Fig. 7. The Scientist-Humanist dichotomy in terms of the Despise&Stigmatize pattern.**

In accordance with Figure (7), I venture

**Opinion 6:** At the root of today's educational dilemma is the fact that the *engineer* of old has become the *enginerd* of today - and nerds just aren't sexy enough to work as role-models for young people.

As pointed out above, the word *nerd* is a rather modern linguistic invention, which encapsulates almost everything that is totally non-sexy. Hence, it was not invented back in the 'glorious old days' of the engineer, when it would have been utterly unthinkable in its present connotation, since it would have conflicted with the underlying role-model. In fact, the emergence of the word 'nerd' constitutes in itself a form of *linguistic reflection* of this role-model transformation. I have taught and researched in mathematics and computer science at the Royal Institute of Technology (KTH) since 1967. During this time I have witnessed the gradual transformation of the classical role-model of the 'engineer', a type of silent, problem-solving, social hero, into the present 'enginerd', a socially ridiculed and ostracized type of person that never gets laid and has severe problems with handling basic aspects of everyday reality, like buying decent clothes and handling bad breath.

Moreover, it seems to me that

**Opinion 7:** The computer represents a kind of bridge between the nerdy and the cool. Basically, the more you know about how to make it work - the nerdier you are, and the more you use it just to ‘do your own creative stuff’ (like music, art or poetry) - the cooler (or hotter) you are.

Therefore, I think that

**Opinion 8:** The computer has the potential to become a useful ally in the *de-nerdification process* that is necessary in order to reinstall scientific interests in the minds of the young.

# 6 The Educational Crisis - The Carrot Rape

## 6.1 The Educational Disfunctionality

Today most nations acknowledge that there is something fundamentally wrong with their school-system. In almost every democratic country in the world this opinion has ventured to the forefront. During the dawning of the information age - at a time when education is globally recognized as the most important strategic investment in the future - more and more young people are failing to absorb the ‘necessary knowledge’ in the form presented by the country’s school systems. The performance problems seem to appear on all levels of the educational system, although they are most clearly visible in the compulsory sectors. One is bound to ask: why is this so?

During the ‘old times’ the schools were run on the classical mixture of the *carot* and the *whip* - leaning heavily towards the latter. The ‘classical’ *realskola* (≈*secondary school*) and especially the *gymnasium* (=*highshool*) often took on the form of a mental torture-and-humiliation arena, that was brought down upon you as soon as you were caught not having done your homework and knowing your answers. This kind of school system has been vivdly portrayed in the Swedish film ‘Hets’, and this is what I mean by a school based on the whip. I was educated in the old, letter-graded school system which still carried those basic traits. The thought of coming to school and not being able to answer for all of my home assignments still raises a vivid feeling of terror inside me.

I am greatful that my daughter doesn’t seem to suffer the same kind of anxieties. On the other hand, I cannot help noticing how easy it seems to be for her to find excuses - in order to get off easy - not having to strain her capacities beyond the humble bounds that are expected by her school-teachers.

In short, the schools of today, while still *compulsory*, have been *deprived of the whip* as a pedagogical tool. In my opinion, this is the real reason behind the continuing outbreak of ‘school-reforms’ that has been shaking the educational system ever since the sixties.

The abandoning of the whip was a necessary consequence of the victories of the ‘human-rights’ movement as applied to children; the inevitable outcome of the school ‘democratization’ process. This evolutionary process has resulted in several analogous changes - as e.g. the criminalization of child-spanking.

Hence, due to the gradual evolutionary disappearance of the whip, the school-reforms have all been aimed at ‘polishing up the carrots’ in various ways - in order to make them more palatable to the ‘educational customers’. The problem with this outlook is that many ‘customers’ are not directly driven by their desire to consume - to put it mildly - but rather by their desire to endure - endure an endless torture of uninteresting and repetitious activities that they didn’t ask for in the first place! And on top of that, you have to put up with people that are payed to pretend that it´s all fun and interesting - and seldom dare to confront the fact they are responsible for confining you, since your presence is required by public law.

It’s a bit like being force-fed with carrots, which is why I will refer to this process (educational design-pattern) as the *carrot rape*. An increasing number of young people react to it by abandoning their learning-projects (‘losing their interest’ as the term goes) - often finding refuge in the willingly awaiting arms of the ‘pop-industry’.

In fact, the word ‘techno’ constitutes in itself an illustrative example of this process. I tend to associate ‘techno’ with ‘techno-logic’, which conjures up images of various ‘engineering dreams’ - heroic deeds to the benefit of the progress of mankind that I read and fantazised about as a child. I can still remember the feeling of exitement looking at the coloured drawings of a nuclear reactor or a particle accelerator. They were illlustrations out of a physics book I had received as a ‘premium’. I was 14 at the time, and this was the most delicious ‘nerdie-num-num’ I had ever seen. In fact, it significantly influenced my future by contributing to the background of interest that later made me enlist at KTH to become a ‘civilinjenjör’ and ‘teknisk fysiker’.

For the young people of today, the word ‘techno’ carries totally different connotations. I think it would be safe to predict that most of them would associate ‘techno’ with a certain style of music - often connected to the ‘tech-nologic’ (= ‘tech-lunatic’) of a rave-party. Unfortunately, the way things stand at the moment, a high-tech-nological rave-party can be a lot more engaging than some (nerdy) techno-logical contraption.

It is the transformation from ‘logic’ to ‘nologic’ which is most disconcerting and alarming in the process described above. By consistently disregarding the realities of the present teaching/learning configurations, and focusing on ‘getting the job done’ (with or without the cooperation of the pupils) the carrot-rape has played a decicive role in producing a lost generation of ‘knowledge-seekers’. Thousands upon thousands of young minds that instead of devoting themselves to strengthening their mental faculties, are dedicated to various aspects of the ‘Brick-in-the-Wall philosophy’ so adequately expressed by Pink Floyd: We don’t want no education - we don’t want no thought control![1]

The following ‘gedanken-experiment’ illustrates the present deplorable state of affairs: if the education of art was to be structured like the education of mathematics, it would be all about how to cut stone and mix paint. Not a single picture, to convey a sense of meaing - let alone beauty, would be shown until the student had mastered all the necessary techniques to paint a similar picture himself! In mathematics education we do not generally deal with theorems that we cannot fully explain and ‘prove’ to the students, whatever that is. When it all “comes down to dust”, a mathematical proof is just an argument that is presented in such a way that it can convince the leading experts of the field. Such a proof is often of little or no interest to the student - especially during the early stages of his contact with the corresponding mathematical theorem. What you need in order to promote your understanding is help to conceptualize the situation that is described by the theorem - to be able to paint your own mental pictures of what is going on. If this situation awakes enough of your curiosity, then you might be interested in finding out why things works the way the theorem describes. Then is the time to provide you with a proof, or - even better - for you to work out your own!

1. Pink Floyd, *Brick in the Wall*, 1977. To illustrate the absurdity of the present attitude towards mental knowledge, just imagine the corresponding outcry in the material domain: ‘We dont want no physical education, we don’t want no muscular control.’ The present ‘gym-hysteria’ is proof enough that such an outcry would find it hard to attract many followers.

This, I believe, illustrates one of the most fundamental errors that we commit as mathematical educators: We present you - the student of mathematics - with the *theorems far too 'unproblematically'*, and the *proofs far too early*, with two major negative consequences: First, your *understanding* is diminished since you have not been provided with the adequate time to conceptualize the mathematical prerequisites for the theorem in question.[1]

Second, your *interest* is diminished, since *the period of curiosity is lost.* As a student, you are not given the chance to know *what* is going on - whithout at the same time being told*why* it is going on. If such periods of curiosity are 'well supported' by the educational system, they have the potential to create an intellectual tension - a kind of 'mental itch' - that may get your mind off the ground and eventually launch it into thought-space once and for all!

## 6.2 Educational Design Patterns - School Duties versus School Rights

Many teachers still seem to implement the pattern of the traditional teacher-preacher - discussed in Figure (8). It goes well with the compulsory-learning pattern which is another essential component of the overall pattern of the educational process, especially in its early parts. It also reinforces the employment-security attitude that is inherent in the tenure-based pattern of permanent teaching positions. Getting a permanent staff position (getting your tenure) is still considered to be the main measure of qualification within the teaching community.

At the same time, the Teacher-Tenure/Learner-Duty pattern of Figure (8) is consistent with the following attutude: Since the learners have to listen to you anyway, you might as well tell them a thing or two that you think they should know about - whether they like it or not. This may in fact be one of its major advantages as opposed to the purely interest driven ConsultingTeacher-Resource pattern of Figure (9). Sometimes you need a push to get you through some 'dull' part of your journey towards knowledge. How to supply such pushes is a problem that is associated with any purely interest-driven educational system.

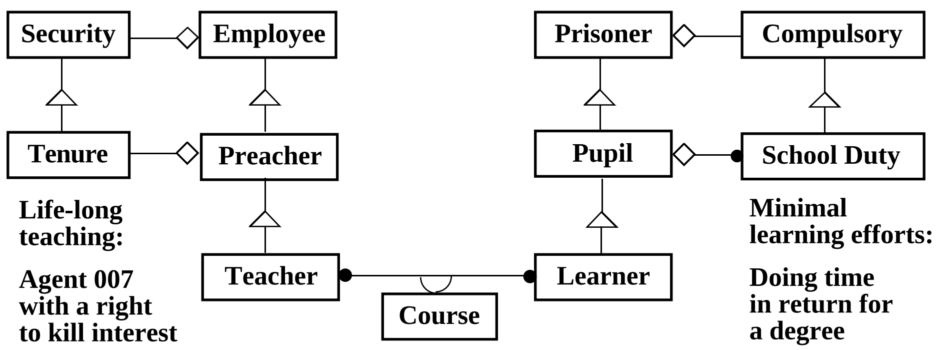


**Fig. 8. The Teacher-Tenure / Learner-Duty educational pattern.**

1. This leads to a tendency to *memorize* the proof, just as with any algorithm.

Interpretation of Figure (8):This is the traditional pattern of *life-long teaching* and *compulsory learning* which is practiced in most forms of earlier education of today. The *teacher* is seen as a *tenured* (= *securely employed*) *preachure*, who can teach as he or she wants to without having to worry about whether the *learners* like it or not - at least not as long as their signs of dislike can be kept within socially controllable bounds. This role-model is referred to as *'agent 007 with a right to kill interest'*.
The *learners* are seen as *pupils* with a number of *school duties* which makes them *prisoners* of a *compulsory* detention system. In this way they adopt a strategy of *minimal learning efforts - doing their time in return for a degree.*

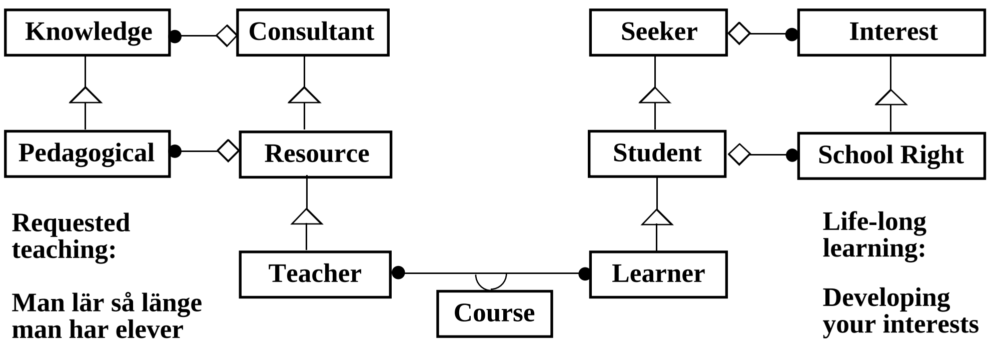


**Fig. 9. The Teacher-Resource / Learner-Right educational pattern.**

Interpretation of Figure (9): This is the emerging educational pattern that characterizes much of the present course-development for industry, as well as the many-fold activities of various kinds of 'study-circles'. The *teaching* is not life-long, but rather performed *on request.* The *teacher* is seen as a *pedagogical resource* which is a form of *knowledge consultant*. When somebody(ies) are interested to learn, then is the time to teach. *Man lär så länge man har elever!* (≈ You teach as long as somebody is learning.)
On the complementary side of the pattern, the *learners* are seen as *students* with a number of *school rights*, a kind of knowledge *seekers* basing their studies on *interest.*

The effective implementation of educational systems founded on the pattern of Figure (9), will demand different ways of thinking about the role of the teacher. In the educational systems of tomorrow, the switch-boarding possibilities of cyberspace will used in order to connect knowledge-sources of great quality with learners of great quantity - in order to promote knowledge-transfer strong interest and high strong. Within such an environment, the role of the traditional teacher can be split up in - at least - a three-fold of different aspects, according to the pattern presented in Figure (10). Here the functioning of the teacher is split up into the three different roles of *teacher-preacher*, *teacher-gardener* and *teacher-plumber.*

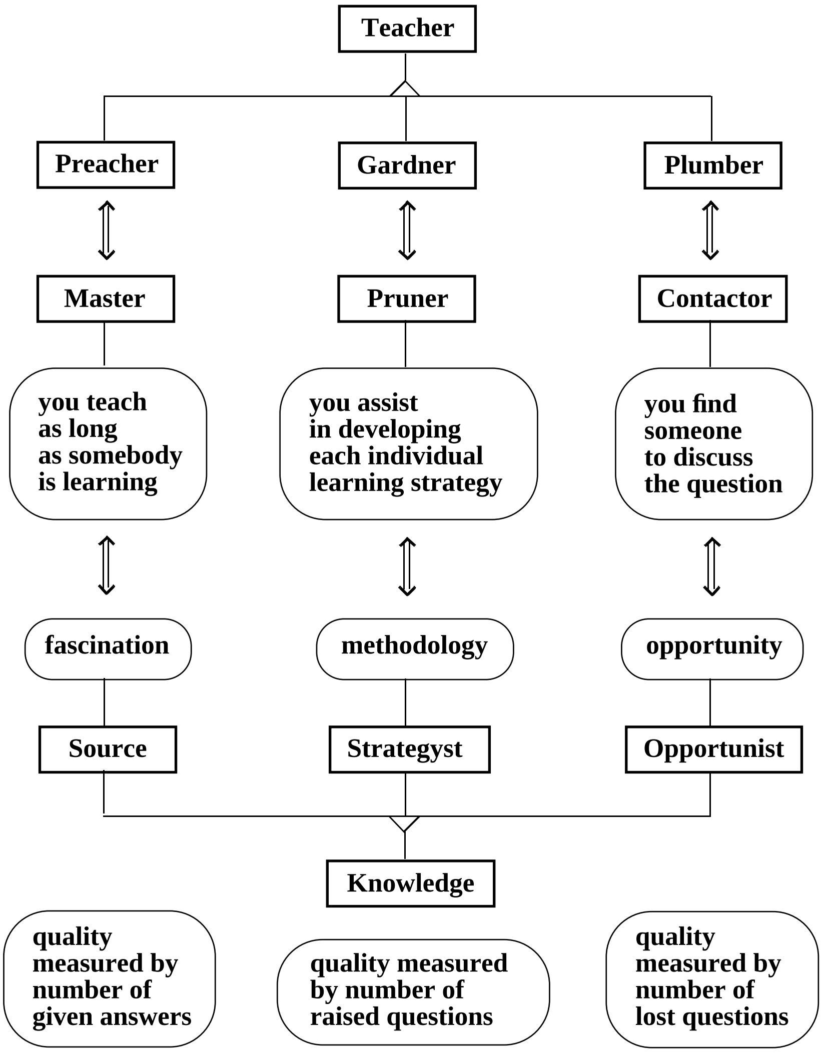


**Fig. 10. The Preacher/Gardner/Plumber KindOf Teacher.**

<u>Interpretation of Figure (10):</u> The Teacher-Preacher/Gardner/Plumber pattern expresses the three different dimensions of teaching that correspond to the Teacher-Resource/Learner-Right pattern of Figure (9).
The *teacher-preacher* is still in the picture, but now as a hired prophet (= consultant) who is a *master* of some kind of *knowledge* that is interesting to others. This knowledge source is driven by fascination and teaches as long as somebody else is learning. The quality is measured in terms of the number of given answers.
The *teacher-gardner* is the *pruner*, who nurtures and prunes the growing knowledge of the learners. The teacher-gardner *assists in developing each learner's own individual learning strategy.* As a *knowledge strategyst*, the teacher-gardner concentrates on *methodology* and is measured - in terms of *performance quality* - by the *number of raised questions*.
The *teacher-plumber* represents the contact with the external world. Whenever a learner question is unsatisfactorily processed, the *t*eacher-plumber *finds someone to discuss the question*. This 'someone' is most often another source of knowledge that was not known - or available -

to the corresponding teacher-gardner. In this way the teacher-plumber creates *opportunity* for learning, and is therefore described as a *knowledge opportunist*. The *performance quality* of a teacher-plumber is measured in terms of the *number of lost questions*.

### 6.3 Examination versus Insamination

“Vi är lika präglade på skolan som gässlingarna på gåsen”[1], - according to Bodil Jönsson. One of the things that we are most ‘hung up on’ is the systems of examination.

Every experienced teacher knows that the main pedagogical problem of a classroom teaching configuration is caused by the differences of intellectual background among the students. In fact, it is a large part of the classical teaching challenge to be able to handle these differences with as little negative impact as possible on the quality of the course. The problem of differences in states of preparation increases in size and importance through every step of the educational system, since they operate with a cumulative effect: what one didn’t understand during previous courses makes it impossible to comprehend what is going on in the present one.

The quantitative differences inherent in the basic grading system have developed from 7 (letter-based grades), through 5 (relative grades) down to 4 (knowledge-related grades). At the same time the ‘through-put pressure’ has increased by at least an order of magnitude. This has resulted in the well-known ‘grade-inflation’ that has accompanied the corresponding ‘weakening of the whip’ as a tool of pedagogical importance. The old letter-based grades were pretty stable in their role of solid ‘educational currency’. However, after the paradigm-shift in favour of the ‘relative grades’ that took place during the sixties (1963-69), the grade-inflation picked up[2]. The effect was to drive the system into the wall and short-circuit its dynamic range. This effect became visible in the seventies, when courses began to appear with entrance requirements of 5.0 grade average - while at the same time 5 being the highest possible grade obtainable within the relative system. The de facto educational configuration was in fact conveying the following message: If you plan to succeed in entering this course or program, you must not be ‘dynamically gradable’, i.e. your grade must not fall within the dynamical range of our grade-system.

The dream of ‘studentexamen for my children’ - ‘the white cap (‘den vita mössan’) on their heads’ - has been a decisive factor in shaping the succession of post-war educational reforms within the Swedish school-system . When I graduated from high-school in 1966, I passed the ‘studentexamen’ in the old, traditional, letter-graded way. The concept of your ‘passing’, as well as the level of your grades, were decided in cooperation between your subject-teachers and a special group of so called ‘censors’ - that arranged special interrogation procedures to which

1. “We are just as hung up on the structure of our school system as the gooslings on the mother goose”.
2. This historical development carries strong similarities to the corresponding evolution of economics. In the old ‘BrettonWoods-period’ of 1944-71, prices were stable and inflation almost non-existant. But when Nixon ‘closed the golden window’, and abolished the fixed exchange-rate between the dollar and the gold back in 1971, he built inflation and other types of chain-letter-like instabilities into the economic system itself. In taking this historic decision he did, however, accomplish much more than that. In fact, he created the logical prerequisites for a totally different type of economy, the so called ‘finacial economy’, which is the totally dominating form of economic activity in the world of today. In the trans-national sector, the ‘financial economy’ outnumbers the so called ‘real economy’ (the trading of goods and services) by a factor of about 50 to 1. See e.g. Kurtzman, *The Death of Money*, [(93)].

you were subjected during your last day at school. These interrogations were by no means only formal procedures, devoid of ‘dangerous possibilities’. Of course, most of the pupils ended up passing - but some of them did not. They were ‘kuggade’ (= ‘rejected’) and kicked out of school after 3 years of study without any form of final diploma at all! In fact, this is what happened to an old class-mate of mine - who, by the way, is now a quite sucessful engineer!

As an example of the reformation of the high-school curriculum, let me mention the old ‘teknisk linje’ (for the ‘engineers’), which has evolved and branched out into something like the ‘fordonsteknisk linje’ (for the ‘machos’), the ‘mediateknisk linje’ (for the ‘trendies’), the ‘vårdteknisk linje’ (for the ‘softies’), the ‘naturvårdsteknisk linje’ (for the ‘greenies’), and the ‘hårvårdsteknisk linje’ (for the ‘pussy-cats’ and the ‘gays’). And while performing this amazing transformational feat, it has remained the old ‘teknisk linje’ (for the ‘nerds’)[1].

It is a sad and chilling fact that for the last few years, the Swedish high-school educational system has been demanding a higher level of theoretical preparation (= better grades) from a hairdressing apprentice than from an apprentice of science or engineering. In my opinion, this is another alarm-signal which indicates the serious malfunctioning of both the grading-system (in particular) and the educational system (as a whole).

Nowadays, the choice of educational program is more interest-driven than ever before. Young people choose their education more and more out of interest, and less and less out of strategic decisions regarding their future careers. And - lets face it - it is simply more popular to become a hair-dresser, than to become a scienctist or an engineer! As I stated in the introduction, during the three decades of my own educational activities, I have witnessed the gradual transformation of the classical *engineer*, into the present *enginerd.*

The latest grade-reform, (LP-94) has introduced so called ‘kunskaps-relaterade betyg’ (knowledge-oriented grades). They consist of 4 different formal levels, ‘ÄnnuEjGodkänd’, ‘Godkänd’, ‘VälGodkänd’ and ‘MycketVälGodkänd’, as well as one ‘informal’ level called ‘streck’, which means ‘not gradable’. The latter grade is not supposed to be given at all (in theory), but is the one that seems to be increasing most (in practice).

The new ‘knowledge-oriented’ grade system has been hailed by its advocates as an antidote to some of the ‘relative perversities’ of the old (relative) grade system. However, due to the absence of a well-structured, centrally supported knowledge-specification scheme, the different schools have been instructed to (= given the freedom to) supply their own interpretations as to what is required in order to receive the corresponding level of grade. The effects of this type of decentralization process (‘kommunaliseringsprocess’) are especially desastrous to the upper parts of the grade-system, the distinction between ‘VälGodkänd’ and ‘MycketVälGodkänd’, where almost no guidance is supplied by the central authorities. This opens up a new kind of ‘grading-relativity’, where the different schools compete for the brightest and most easily manageble pupils by supplying the highest grade-level per input work-hour. This creates a ‘win-win situation’, where the pupils get their educational advancement breaks while the school earns some easy revenue. The overall looser is the ‘quality of content’, that vital and seemingly non-measurable entity which real education is all about.

1. The psychological ‘targeting’ (= the ‘målgruppsspecifikation’) of these respective course-programs is meant to be taken non-seriously (= as a joke).

To put it bluntly, in the schools of today, the grade-systems are rapidly de-grading into a competitive advantage in the process of attracting students. Hence it is only natural that companies have started to look less at the examination grades of their candidates for employment, and more on their ‘insamination abilities’, i.e. their ability to display enough skills in some form of ‘entrance test’ (*insamination*). It often takes the form of an ‘exjobb’ or some kind of ‘projektanställning’ to ‘show us what you can do’.

So, what can be done about the present examination problem? It seems to me that the natural thing to do is for the schools and universities to go along with the leading employers, sharpening their entrance tests (insaminations) to the courses, while at the same time weakening their exit-tests (examinations) from the courses. It should be considered a merit just to have been accepted into a course, since this means that you have passed its thorough insaminations. Teaching such a ‘well-insaminated course’ would seem like a paradise-dream to most of the teachers of today. Just imagine - a whole classroom full of interested students that are well prepared for the course!

Of course, such insamination structures would require a lot closer cooperation between the different teachers and institutions involved in designing the curriculum, forcing them to deal with how their own particular piece of the story connects with others - thereby fitting it into the larger picture. This is probably the main obstacle to be overcome in such an attempt, since it conflicts with the institutional turf-wars that are only far too frequent within the academic world.

### 6.4 The Structure of Present Mathematics Education

The present mathematics education is heavily influenced by the *Problem/Solution* pattern. This pattern appears within the educational process in at least three different ways - as illustrated below in Figures (11), (12) and (13).

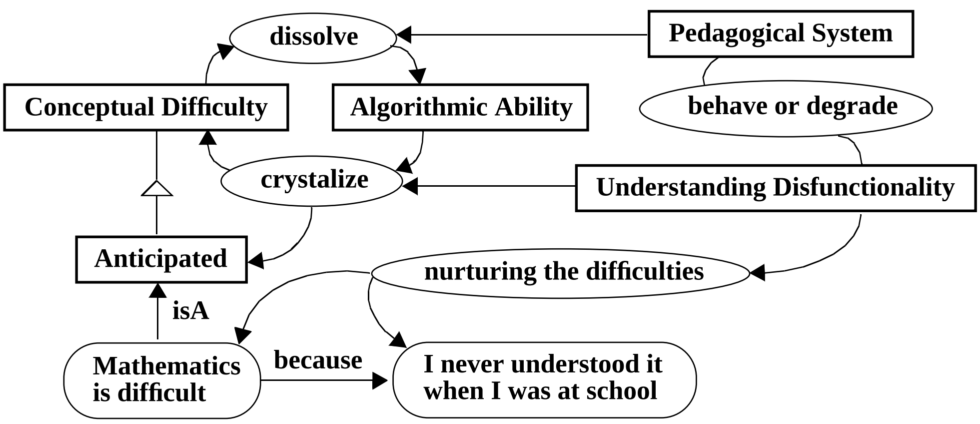


**Fig. 11. The Problem/Solution pattern applied to early mathematics education.**

<u>Interpretation of Figure (11):</u> The pattern describes how *conceptual difficulties* (= concept-formation-problems) in mathematics education are ‘solved’ by promoting *algorithmic ability*, i.e. by teaching various forms of ‘arithmetic schemes’ in a more or less fundamentalist fashion.

This dissolution process is seen as driven by a *pedagogical system* that operates according to the principle of *behave or degrade*, i.e. the traditional ‘math-test-metric’. This leads to severe forms of *understanding disfunctionality*, which in turn drives the *crystalisation* of the *conceptual difficulties*. By nurturing the difficulties the loop creates *anticipated conceptual difficulties*, which are summarized in the biggest mental block of all - *mathematics is difficult*. This conclusion is here seen as an effect of the (be)cause: *I never understood it when I was at school*. This is a natural form of defense-reaction that ties in with the pattern of Figure (14), which describes the filtering of the educational system and expresses the view that “when you do not understand during the later stages of your education, you often end up teaching in the earlier stages”.

In fact, the early stages of mathematics education are performed according to the prevailing pedagogical version of the well-known *propaganda principle*:

“If you repeat a lie a large enough number of times it becomes true” .

The version of this pattern that is applied to early mathematics education could be formulated:

“If you repeat a calculation a large enough number of times, you learn it” [1].

The pattern of Figure (11), which is characteristic of earlier education, has two complementary correspondances - one within the *computational industry* [Figure (12)] and one within *mathematical research* [Figure (13)]. These three versions of the problem-solution pattern cooperate in reinforcing the opinion that “mathematics is difficult”.

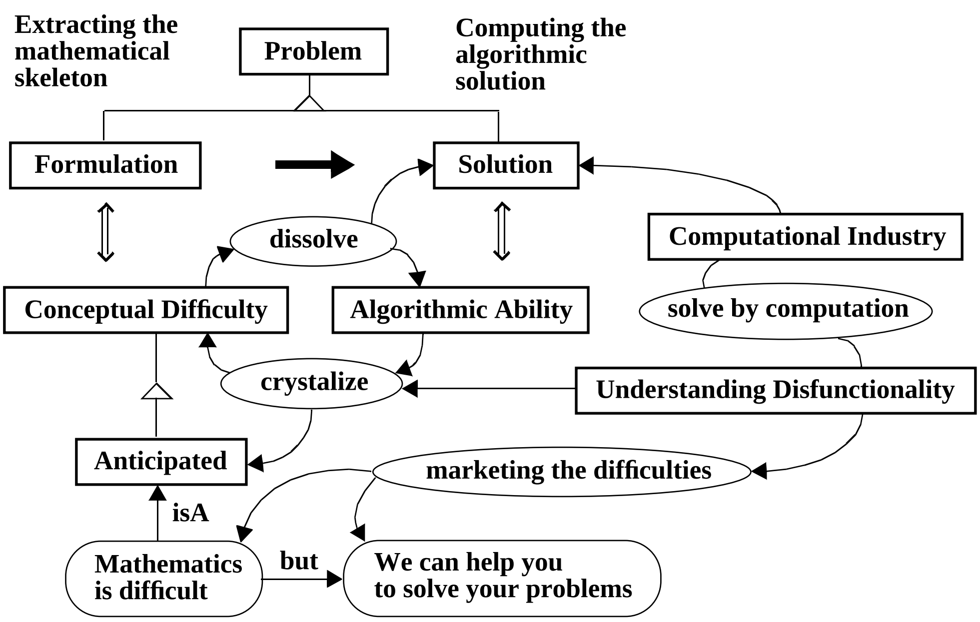


**Fig. 12. The Problem/Solution pattern applied to the computational industry.**

1. i.e. you learn to *repeat* it.

Interpretation of Figure (12): The pattern of Figure (11) is here supported by a similar pattern, involving the *computational industry*. Again, *conceptual difficulties* are *dissolved by computation*, which focuses the mathematical activites on *problem solution* (= computing algorithmic solutions), and away from *problem formulation* (= extracting the mathematical skeleton). This conceptual confusion creates the *understanding disfunctionality* which helps to re-*crystalize* the *conceptual difficulties*. By *marketing the difficulties*, the *anticipated conceptual difficulty* of mathematics itself is firmly established: *Mathematics is difficult, but we can help you to solve your problems.*

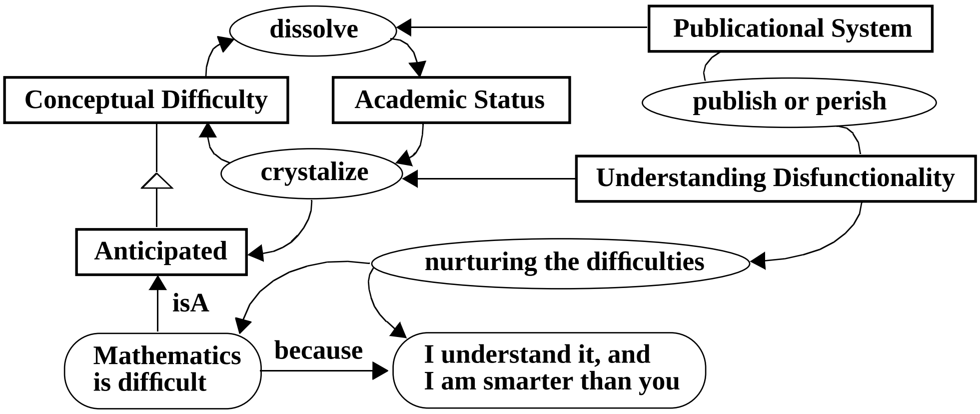


**Fig. 13. The Problem/Solution pattern applied to mathematics research.**

Interpretation of Figure (13): A third variation on the problem-solution pattern appears in *mathematical research*. This pattern shows the *conceptual difficulties* being *dissolved* into *academic status*, driven by a *publicational system* that operates according to the well-known principle of *publish or perish*. This leads to another kind of *understanding disfunctionality* - where research articles in mathematics are written - not in order to be genuinely understood, but rather in order to ‘pee in an academic habitat’, which means to ‘fend of intruders’ and stake an intellectual claim which is as large as possible. This leads to the *nurturing of difficulties* for yet another reason - namely in order to maintain the status of the professional mathematician: *Mathematics is difficult - because - I understand it, and I am smarter than you*.

The Discriminator-Questions pattern of Figure (14) has a profound impact on the educational situation in general, but its effects are probably most pronounced in mathematics. The aspect of the pattern that concerns this issue is the following: *People who did not understand during their later education in mathematics* (= *higher courses*) *are recruited to teach the earlier parts*.

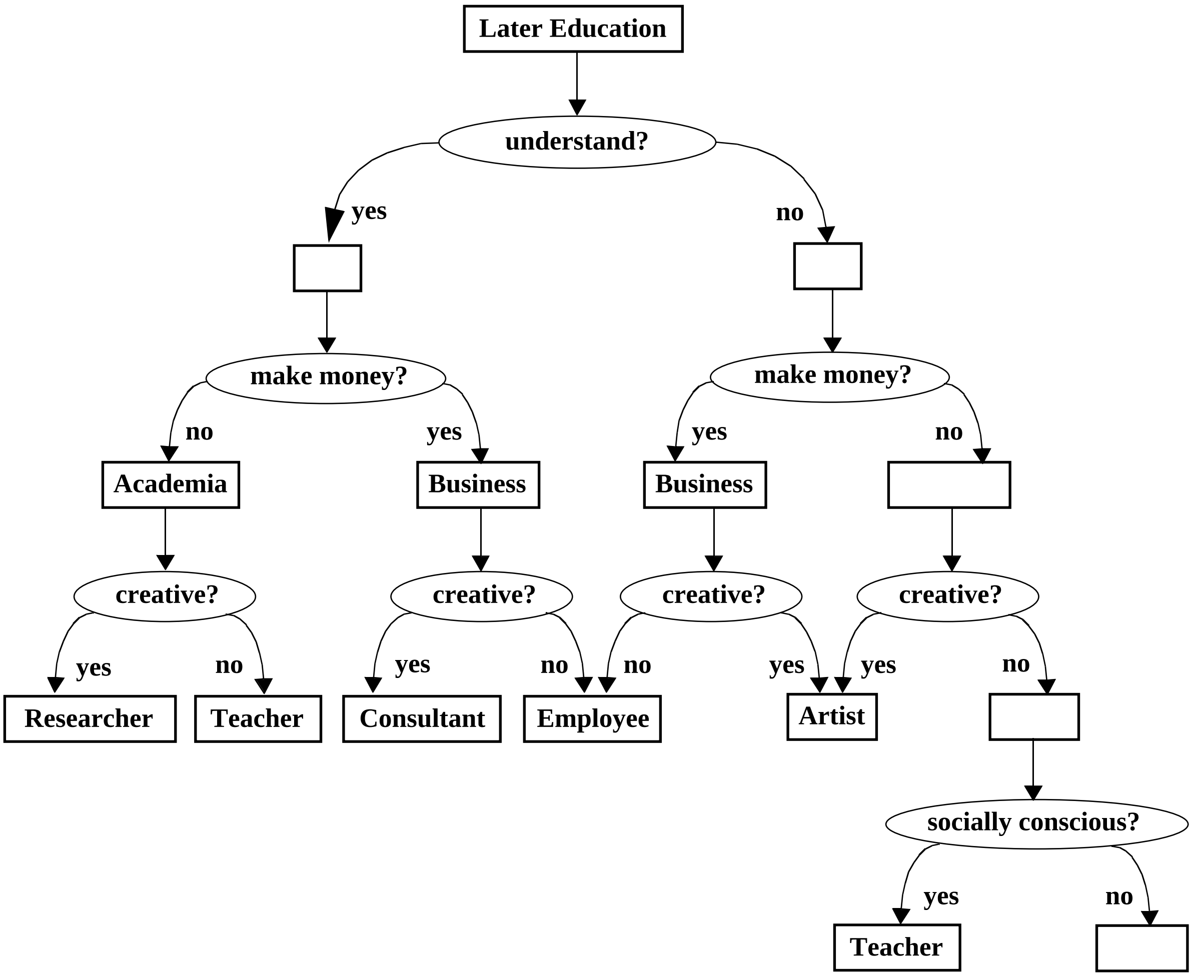


**Fig. 14. The Discriminator-Questions pattern.**

Interpretation of Figure (14): This pattern shows the filtering influence of a series of *discriminator-questions* that confront people that participate in the later (= higher) part of the educational process. The first discriminator is given by the question:*'Do you understand?'*. If the answer is 'yes', the next discriminator-question is *'Do you want to make money?'*. If you answer this question by 'no', then you probably go into some form of *Academia*, where you are confronted with the third discriminator-question: *'Are you creative?'*. If your answer to this question is 'yes', you become a *researcher*, otherwise you become a (academic) *teacher*. This is the pattern that operates behind the well-known academic contempt that the successful researcher has for the sucessful pedagog. Within academia, if you are a good teacher, you are considered by default to be a bad researcher - until you have delivered evidence to the contrary. Going back to the second question, if you did *understand* your later education, but you do want to *make money*, then you go into *business*. In this case, if you are *creative*, you become a *consultant* (≈ inventor) and market your skills in various ways, whereas if you are not, you become an *employee*. In both cases you work as a *specialist* (= dealing with special problems) - the difference is mainly the form (= security) of employment: F-skatt for the consultant, A-skatt for the employed specialist.

Going back to the first question, if you *didn't understand* your later education, but you do want to *make money*, then you also go into *business*. If you are *not creative*, you get *employed* as a *rutinist* (≈ clerk). This is a person that takes part in operating the 'everyday schedule of things' - as opposed to a specialist, that takes care of 'special situations'.
If, on the other hand you are *creative*, you become an *artist*, and often a *commersially sucessful* artist too. Going back one level, if you *didn't want to make money*, but you are still *creative*, then you also become an *artist*, but in this case you are not so commercially successful as before. Instead you 'realize your inner artistic potential' in various ways, often enduring substantial forms of economic hardships on the way.
Finally, if you *didn't understand* your higher education, *didn't want to make money* and *weren't creative*, then you are subjected to the fourth question: *'Are you socially conscious?'*. If your answer is *'yes'*, then you become a *teacher* in the *earlier* parts of the educational process.

**Opinion 9:** The Discriminator-Questions pattern of Figure (14) has a bearing on *why* early teaching has become so dominated by women. The social form of intelligence[1] displayed in the traditional female role-model is very lowly valued by the economic society, which is reflected in the low salaries associated with early teaching.

An important aspect of the Discriminator-Questions pattern is the *order* in which the questions are asked. In the pattern of Figure(14), the question concerning social intelligence is the one that is asked last, which reflects its lesser degree of estimated importance. Since there are four different questions represented, there is 24 different ways to order them. Each one gives a different filter of discrimination. It is an interesting exercise to play around with the order of these questions and reflect a bit on the corresponding labeling of categories.

In connection with the all too familiar subject of 'contempt for teaching', I am reminded of the following statement from one of the recipients of the 1983 Right Livelihood Award. In his acceptance-speach he declared - with a glimpse in his eye - that:

> There are three kinds of people. First, there are those who *know*. And then there are those who don't know, and they *teach*. And then there are those who don't know how to teach, and they *teach education*.:

Although he ment it as a joke, there is an underlying aspect that has to be taken seriously in any non-trivial discussion of the performance of the educational system as a whole.

1. See e.g. Howard Gardner's discussion on the different dimensions of intelligence .

# 7 First Class Mathematics

## 7.1 What is Mathematics?

The word ‘mathematics’ is said to go back to Pythagoras, who called his most advanced disciples ‘mathematikoi’ (μαθηματικη ).[1] In the present context, this word will be interpreted in the following way:

**Definition 4:** *Mathematics* is the study of the totality of structures that the human brain is able to perceive: *Mathematics* = Hom(*Universe*, *Brain*).

Within the langugage of itself, mathematics can be described as the study of ‘homomorphisms’ between the brain and its environment - including the brain itself[2]. Life is structure, and since mathematics is the language of structure, it is the language of life, the ultimate ruler of the subspace within which life can be talked and reasoned about.

**Opinion 10:** Mathematics can never be taught.
It can only be given the opportunity to grow.

Mathematics is a feeling, a sensitivity and an awareness of structure, which is planted in each and every one of us, like a seed of the cosmic consciousness. To practice the art of mathematics is to be involved in a purely mental process, which has surprisingly strong connections to the surrounding physical reality.[3]

When one is developing ones mathematical understanding, both halves of the brain work together in the process of constructing combinations of mental phantasies that are tested for logical functionality. The right half of the brain is phantasizing, and the left part is analyzing and testing the logic of the suggested ideas. Only the ideas that survive the logical examination are elevated to the status of truths. Mathematics can therefore be described as *logically tested fantasies*. It offers a powerful means for its devotees to overcome some of their sensous limitations and contemplate the inner profoundity of the structure of the universe.

Nowadays, the preferred style of presenting mathematics is in the form of *theories*. Each mathematical theory consists of *axioms*, *definitions*, *theorems* and *proofs.*

**Definition 5:** The mathematical *axioms* are the ‘initial truths’ - the ‘holy syntax’ which expresses a belief that cannot be questioned or analyzed within the theory itself.

**Definition 6:** A *model* of a theory fills its axioms (= basic syntax) with *meaning* (= semantics) by providing an example where *the axioms of the theory express an actual behaviour pattern of the elements of the model.*

---

1. Heath, *History of Greek Mathematics*, [(68), Vol I, p.11].
2. one’s *self-image*, as it is often called.
3. See e.g. Wigner, *On the unreasonable effectiveness of mathematics in the physical sciences*, [(171)].

**Definition 7:** The mathematical *definitions* introduce new concepts as formal combinations of the old ones.

A good definition packages a modular thought-pattern into a single concept with a crisply defined interface to the others. This supports the *conceptual calibration process* that is necessary to increase the precision of communication, as well as to counteract the onset of combinatorial complexity.

**Definition 8:** A ‘mathematical truth’ is called a *theorem* if it is considered important enough to be an end in itself, a *lemma* if it is considered as a means to an end, and a *corollary* if it is considered to be an obvious consequence of some other theorem(s).

**Fact 2:** A mathematical *theorem* always has the form *A*  *B*, which reads: If *A* is true, so is *B*.

The mathematical term is: *A implies B*, and the arrow ‘ ’ is called the (logical) *implication operator.* The statement *A*  *B* says nothing about the truth of either *A* or *B*, it only states a *relation* between the factual conditions under which *A* respectively *B* are true: the former situation cannot occur without the latter occuring simultaneously. This is why Bertrand Russel referred to mathematics as:

> The science where we neither know what we are talking about, nor whether what we say is true or not.

**Definition 9:** A mathematical *proof* is a logical *chain of reasoning steps* that satisfies the demand of rigour of the leading experts of the corresponding field. The ideal is to carry out a proof completely within the formal system, or at least to become convinced that this kind of formalization could - in principle - be achieved!

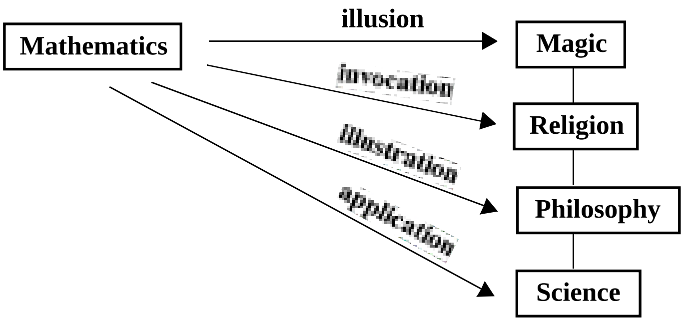


**Fig. 15. Four different levels of mathematical usage.**

<u>Interpretation of Figure (15):</u> There are four natural ‘ways of reasoning’ - that correspond to *different ways to reason about things* and *different things to reason about*. They are called respectively *Science, Philosophy, Religion* and *Magic*. Taken in that order they form a sequence of ‘successive loosening of constraints’. (Figure (15)). In *science* , all concepts are strictly

defined and narrowed down by complicated calibration protocols. In *philosophy*, these protocols are substantially ‘loosened’, and each person is given the right to ponder the deeper questions of life in a more speculative way. *Religion* introduces a new element - *faith* - a concept that philosophy tries to avoid at all costs. This makes it possible to handle a totally different set of questions by *having faith* in various concepts connected to them. If we please our God(s) then (s)he (they) will create some good magic for us and protect us from the bad magic that surrounds us. At the top, *magic* represents the final state of constraint-loosening - where we are victims to the ultimate illusion: the magical trick of ‘anything goes’.

In accordance with these four levels of thinking, there are the corresponding four different levels of *using mathematics*: mathematics can be used for *scientific applications*, *philosophical illustrations*, *religious invocations* and *magical illusions*. As illustrated in Figure (15):

**Definition 10:** The usage of mathematics in *science* is considered to represent an *application*, in *philosophy* an *illustration*, in *religion* an *invocation* and in *magic* an *illusion*.

## 7.2 The First Class Mathematics Project

The *First Class Mathematics* Project aims to convey an image of mathematics as such a ‘double-brained’ mental activity as has been described above. It is of the utmost importance that all students are encouraged and supported in their attempts to develop their own mathematical fantasies. This is only possible if they are confronted with interesting examples of ‘good mathematics’ during every stage of their mathematical education. An illustrative example of what such a confrontation can imply is given by the so called ‘Rubrik´s Cube’ or just ‘the cube’, which was such a fascinating mental torment to the kids a couple of years ago. They got into doing their own advanced forms of algorithmic mathematics - but only during their breaks and ‘free-time’ from a school-system that was totally incapable to realize what was going on, let alone to make use of it in order to promote the mathematical interest of the students.

By the aid of modern computers, many exiting mathematical structures can be animated and brought to ‘interactive life’ in ways which open up new and exiting pedagogical possibilities. Today there is a large number of interactive mathematical tools that empower a student to explore mathematical concepts in a dialectical way, i.e. in a dialogue with the computer program.

## 7.3 The Concept of Symmetry

The feeling of esthetic harmony in a geometric shape or a pattern has to do with its particular mixture of regularity and irregularity. In ordinary language we refer to the regularity of an object as its *symmetry*. When we talk about e.g. ‘the symmetry of a face’, we mean that there is a plane that can be imagined right down the middle of the nose, such that if one carried out a reflection in this plane, one part of the face would reflect into the other part, while the face as a whole would remain (approximately) invariant. These ideas form the basis for a mathematical definition of the concept of symmetry:

**Definition 11:** Symmetry is *invariance* under some form of *change* (= *transformation*).

In their book *Fearful Symmetry - is God a Geometer*, Ian Stewart and Martin Golubitsky, carry out a very enlightening discussion on the subject of symmetry:[1]

> The great physicist Pierre Curie is best remembered for his work, with his wife Marie, on radioactivity, leading to the discovery of the elements radium and polonium. But Curie is also remembered for his realization that many physical processes are governed by principles of symmetry. In 1894, in the Journal de Physique Théorique et Appliqué, Curie gave two logically equivalent statements of a general principle from the folklore of mathematical physics:

**Principle 1:** • If certain causes produce certain effects,
then the symmetries of the causes reappear in the effects produced.
• If certain effects reveal a certain asymmetry,
then this asymmetry will be reflected in the causes that give rise to them.

These two statements usually go by the name of *Curie's principle*. They are often referred to in the popular phrase *'Dissymmetry makes the phenomenon'*. But under certain conditions, symmetric causes can have asymmetric effects, which directly contradicts this celebrated principle of Curie. As an example of this, Stewart and Golubitsky discuss the shape of a splash:

> Our favourite oddball science book is *On Growth and Form* by d'Arcy Thompson. If you've never read this provocative and penetrating treatise, get a copy from somewhere - though be warned, part of its appeal is an outmoded charm, so don't take it too seriously. Thompson was a pioneer of the idea that there are mathematical features to biological form. Prominently displayed at the very front of his book there is a wonderful and slightly disturbing picture of a drop of milk hitting the surface of a bowl, filled with the same liquid; the splash is frozen by high-speed photography, for us to contemplate at leisure. When raindrops hit a puddle, or inkblots hit paper, they must do something similar. Have you ever wondered what shape a splash is? It looks like a crown.
>
> From the point of impact rises a smooth, circular ring, surprisingly thin-walled, curving gracefully outwards as it rises. But the ring doesn't remain circular: it breaks up into 24 pointed spikes. Why does it break up? Why 24? These are good questions. The spikes are (almost) regularly placed. Why? That's another good question. The spikes come to a sharp point; most have just thrown off a tiny rounded droplet of milk (why?), and the rest are about to. [...].
>
> Focus your attention on the symmetry of the splash in d'Arcy Thompson's picture. It isn't perfect, but presumably that's due to slight imperfections in the shape of the original drop or the angle at which it fell. Maybe it was wobbling a little, maybe the milk in the bowl wasn't completely still. But the dominant feature, the spiky crown, doesn't look as though it's caused by such imperfections. You get the feeling that a *perfectly* spherical droplet would just give a *perfect* (and very probably also 24-pointed) crown!

Without going into detail, it turns out that with the right interpretation Curie was right all along, but many of the consequences people draw from his principle are wrong, because they use the wrong interpretation. According to Stewart and Golubitsky:

> Curie was right in asserting that symmetric systems have symmetric states - but he failed to address their *stability*. If a symmetric state becomes unstable, the system will do something else - and that something else need not be equally symmetric.
>
> This is puzzling. A slightly rotated splash, with its spikes where once there were gaps, seems to be just as valid an effect as the splash that actually occurs. Can the droplet have *more than one effect* ?

---

1. See [(156)], preface, p. xviii.

In the real world, no: something definite has to happen. You don't get two splashes at the same time. But in mathematics, yes. Both splashes, the original and its rotation, are valid solutions to the same equations; valid consequences of the same physical laws. Instead of a single effect, we have a whole *set* of possible effects: all the different rotations of the 24-pointed crown. The logical fallacy is the assumption that each cause produces a unique effect. When the system becomes unstable, instead of a single effect, *a certain cause may have a whole set of possible effects.*

This paradox - that symmetry can get lost between cause and effect is called *symmetry breaking*. In recent years scientists and mathematicians have begun to realize that it plays a major role in the formation of patterns. From the smallest scales to the largest, many of nature's patterns are a result of broken symmetry. According to Stewart and Golubitsky: [1]

History is littered with examples where scientists and philosophers have misapplied Curie's principle, seeking *large-scale* asymmetries in causes, to account for large-scale asymmetries - such as patterns - in effects. For example, until very recently astronomers thought that the spiral arms of galaxies were caused by magnetic fields. Now they're beginning to think that the spirals are the result of gravitational symmetry-breaking. But Curie's principle doesn't say that the size of the asymmetry is comparable in the cause and the effect. In fact, when systems have symmetry, there is a good chance that the symmetry may break. When it does, very tiny asymmetries play a crucial role in selecting the actual outcome from a range of potential outcomes.[2]

The interaction of symmetry with dynamics is itself a rapid growth area of research: it may not be as well known as chaos, but in some respects it's probably more useful. Humanity has always been better at exploiting patterns than it has chaos; though perhaps better at generating chaos than order. It may seem that there is an unbridgeable gulf between symmetry and chaos; but one of the most exiting prospects for future discoveries lies in their interaction. In the dynamic behaviour of systems with symmetry mathematicians have stumbled upon a natural meeting-place for order and chaos.

#### 7.3.1 The Mathemagic of Wallpaper Patterns

This part is concerned with the mathematical structure of *wallpaper patterns*. Surely you have at some occation studied the pattern of some wallpaper and thought about the different ways that the pattern can be shifted (= translated), rotated and reflected in such a way as to be transformed into itself. The set of all such transformation (= isometries) constitute the so called *symmetries* of the corresponding pattern. Togther these transformations - that leave the pattern as a whole unchanged (= *invariant*) form a mathematical structure called a *group*, and therefore one usually talks about the *symmetry group* of the pattern.

The mathematics of wallpaper patterns are not part of any standard mathematics curriculae, neither at the level of secondary school, high school or university. What follows is a summary of a speech on this subject that I have given on several occations.[3] Part of this material has been used by a school class of 10 year old children within the context of the *First Class Mathematics* project discussed below. It has turned out to appeal to both the boys and the girls in the class - and especially to the latter - since it brings out the mathematics of pattern-making - such as e.g. stiching, knitting, etc - which are traditionally female types of activities. In my mind

1. [(156)], p. 17.
2. *Loc. Sit.*, preface, p. xviii.
3. Västerås, March 18, 1995 (the Swedish Association of Mathematicians), Sundsvall, January 26, 1996 (the 9:th Mathematics Biennal), Stockholm, January 25, 1997 (Matematics Biennette 97). As if by some form of synchronicity, the world famous mathematician John Horton Conway turned up in Stockholm about 6 months after my presentation in Västerås and gave an unforgettable lecture on the symmetries of the wallpaper pattern - where he discussed them in terms of the so called 'orbifold notation' (due to Bill Thurston).

**Opinion 11:** The reason that girls seem to display less interest in mathematics than boys has to do with the fact that the school system does not intellectually acknowledge the traditionally female forms of pattern-mathematics. This highly intricate and stimulating form of mathematics is only given tactile and practical acknowledgement in the form of e.g. sewing, knitting and weaving courses.

The geometric structure inherent in various forms of patterns has asserted a strong fascination on the human mind across different times and cultures. A multitude of art and craft - that has been preserved through the ages - bears strong witness to this fact. However, it took to the end of the last century until the underlying mathematical structures in the various forms of patterns could be made precise and find explicit expression in terms of their so called *symmetry.* The mathematical concept of symmetry was, in turn, a logical consequence of the concept of *group* - which is one of the most important structures of modern mathematics, and which evolved into its general (abstract) form during the last century. What follows is a brief overview of how to classify wallpaper patterns with the help of these concepts.

The general abstract definition of a *group* can be formulated thus:

**Definition 12:** A *group* is a *set* with an *associative binary composition*, where there is a *unit element* and where every element is *invertible*.

This definition is extremely compact, and requires an understanding of the concepts of *set*, *associative binary composition*, *unit element* and *invertible* in order to become meaningful. To be more specific, I will exemplify with the type of mathematical group that will concern us below - namely the group of all motions. Two motions can be *combined* (= *composed*) with each other) by being carried out in succession (one following the other), and the resulting combination is always another motion. This phenomenon of creating one motion by combining two others is referred to as *binary composition.*

If three motions - called $a$, $b$, and $c$ - are carried out in succession, their resulting motion is independent of whether we first combine $a$ with $b$, and then combine the result with $c$ - or whether we instead combine $a$ with the combination of $b$ and $c$. Such a binary composition is called *associative*. Moreover, there exists a type of motion that does not affect (= change) any other motion through the means of composition, namely the motion of *rest* (= standing still). This 'zero-motion' functions as the *unit motion* of the group. Finally, to each motion $m$ there is a uniquely associated *inverse* (= *neutralizer*) motion (often called $m^{-1}$), which neutralizes the effects of $m$ by moving everything back to where it came from, so that the combined effect of the two motions is the zero-motion of identical rest.

Consider an ordinary wallpaper pattern. It has the (idealized) properties of being located in a single plane as well as being indefinitely repeatable along the directions of this plane. It is always possible to select certain parts of the pattern and then form the rest by translating (= sliding) the selected part along such directions. If we carry out these translations (= parallel shifts) on the entire (= infinitely repeated) pattern, it is easy to see that this pattern - taken as a whole - will remain unchanged (= invariant). The pattern is 'transformed into itself' - as a mathematician would express the matter.

Apart from such sliding motions (= *translations*) there are often other types of motions whose effects are not noticeable on the pattern as a whole, namely *rotations* (with certain angles around certain points) and *reflections* (in certain lines). Of course, a reflection is not an actual motion (since it reverses *parity*), but if we expand the concept of motion to include each type of transformation that does not change the distance between any pair of points, then we can regard reflection as a generalized type of motion that is called an *isometry.* The actual motions are referred to as *direct* isometries, and they form a *subgroup* of the *group of all isometries*. Reflections belong to the so called *reversing* isometries. They do not form a group under composition, since the composition (= product) of two reversing isometries is direct. In fact, the combined effect of two successive reflections is either a translation or a rotation - depending on whether the two reflecting lines (= mirrors) are parallel or not.

I will now introduce a mathematical definition of the concept of symmetry. Within the use of everyday language, the word ‘symmetry’ means roughly ‘regularity’ or ‘sameness’ in some way. Within mathematics, however, one associates the word ‘symmetry’ with the concept of group by introducing the idea of a symmetry-group (= group of symmetries). This idea is based on the following

**Fact 3:** For any given wallpaper pattern, the set of its non-changing isometries (= the set of isometries that leave it invariant) form a group.

Hence the following definition makes mathematical sense:

**Definition 13:** The group of isometries that leave a certain pattern invariant is called the *symmetry-group* of the pattern.

The fundamental idea - which has elevated the study of symmetric patterns into a mathematical discipline - is to classify the various patterns with respect to their different symmetry-groups.

**Definition 14:** Two patterns belong to the same symmetric family if their respective symmetry-groups are isomorphic.[1]

In order to describe the structure of the symmetry-group of a wallpaper pattern, one can make use of some basic facts from plane geometry, which are listed below for convenience The proofs are simple, but are not given here.

**Fact 4:** A *planar isometry* is uniquely determined by its image of 3 non-collinear points, i.e. by how it maps 3 points that are not all located on the same line.

**Fact 5:** Every *planar isometry* can be expressed as a combination of at most 3 reflections.

**Fact 6:** The combination of 2 reflections is a *translation* if the two lines are parallel, and a *rotation* if the two lines intersect each other.

1. i.e. if they have the same structure (as groups).

**Fact 7:** The combination of 3 reflections
is a *reflection* if the 3 lines are parallel or if they intersect in a common point,
and a *glide-reflection* otherwise.

From these facts one can draw the following conclusion:

**Fact 8:** Each *planar isometry* is either
a *translation*, a *rotation*, a *reflection*, or a *glide-reflection.*

We will see below how each symmetry-group is characterized by its special mixture of these participating isometrical ‘building-blocks’. We begin by studying the structure of the participating rotations. Consider a rotation $r$ that is a member of the symmetry-group of a certain wallpaper pattern. The rotation $r$ is always carried out around some point $P$ in the pattern, and $P$ is referred to as the center (of rotation) for $r$.

**Definition 15:** If a certain rotation $r$ in the symmetry-group of a wallpaper pattern
turns the plane through an angle of $360/n$ degrees,
the centre of rotation of $r$ is said to possess *n-fold rotational symmetry*
with respect to the given pattern.

Moreover, there is the following important:

**Fact 9:** Each centre of rotation for a wallpaper pattern
possesses either 2-, 3-, 4-, or 6-fold rotational symmetry .

**Proof:** We start by selecting two centers of rotation $P$ and $Q$ - located at a minimal distance from each other - and investigate what happens when they are subjected to a rotation of the pattern’s symmetry-group. It is easy to show that every other type of rotational symmetry than the ones listed above must turn $P$ and $Q$ into two new centers of rotation that are even closer together that $P$ and $Q$ . But this amounts to a contradiction, since we have chosen $P$ and $Q$ to be two minimally distant centers of rotation to start with. <<>>

Hence, the only types of rotations that can be part of the symmetry-group of a wallpaper pattern are those that rotate the plane by 180°, 120°, 90°, or 60°. This fact forms the basis for the partition of the wallpaper pattern symmetry-groups that is shown in Figure Y. Each column contains patterns with the same kind of maximal rotational symmetry, i.e. with the same minimal angle of rotation. In order to get on with the classification we can make use of *Sylvester’s theorem*:

**Fact 10:** For a given wallpaper pattern, consider three neigboring centers of rotation $P$, $Q$, $R$ - with $p$-, $q$-, respectively $r$-fold rotational symmetry, where at least two of $p$, $q$, $r$ are greater than 2. Then it follows that
$1/p + 1/q + 1/r = 1$.

**Proof:** The proof is based on the fact that the combination of two rotations is another rotation. Let $a/2$, $b/2$, $c/2$ denote the size of the angles of the triangle $PQR$, and let $P_a$, $Q_b$, $R_c$ denote rotations through the angles $a$, $b$, $c$ around the points $P$, $Q$, $R$ respectively. It follows that the combined motion $P_aQ_bR_c$ must be identical to a rotation around some point through the angle $a+b+c = 360°$ , i.e. $P_aQ_bR_c$ = the zero-motion of rest. Now, since the point $P$ has a $p$-fold rotational symmetry, and since $PQR$ is minimal, it follows that $a = 360°/p$, and by analogous reasoning we get $b = 360°/q$ and $c = 360°/r$. Since $a+b+c = 360°$, we have established that $1/p + 1/q + 1/r = 1$. <<>>

Sylvester's theorem implies a strong limitation on the number of possible wallpaper symmetry-groups. If a pattern has rotations with more than 2-fold centers of symmetry, these must correspond to an integral solution $p$, $q$, $r$ to the above equation $1/p + 1/q + 1/r = 1$. However, it is easy to see that there are only 3 different possibilities, namely (i): 1/4+1/4+1/2, (ii): 1/3+1/3+1/3 och (iii): 1/6+1/3+1/2. These solutions correspond to the three rightmost columns in Figure (16). The two columns to thet left correspond to (iv): no rotations and (v): only rotations with 2-fold symmetry.

These five different cases can now be further analyzed with respect to the possible presense of reflections and glide-reflections. This analysis is rather tedious and is omitted here for lack of space. It leads to the conclusion that there exist a total of 17 different possibilities - which we formulate as

**Fact 11:** There are exactly 17 different types of symmetry-groups for wallpapers.

The reader is referred to [(49)] for a complete proof. The 17 different possibilities are all illustrated in Figure Y, while Figure (17) shows a classical flow-chart, by which one can easily identify the type of symmetry-group of any given wallpaper pattern.

The first proof of the fact that there are exactly 17 different symmetry-groups for wallpaper patterns was given by the Russian mathematician Fedorov in 1891 - curiously enough a few months after he had proved the corresponding theorem in 3-dimensional space. In order to get a compact formulation of this theorem, it is practical to make the following

**Definition 16:** A *crystal* is a pattern which repeats itself endlessly along the three different dimensions of space.

In three dimensions, Fedorov's theorem can now be formulated thus:

**Fact 12:** There are exactly 230 different symmetry-groups for crystals

The proof of this fact is substantially more complicated than the corresponding proof in the planar case of wallpapers. A complete proof is given e.g. in [(150)]. It is important to point out that all of the 17 different wallpaper symmetry-groups were discovered empirically a long time ago - and have been used by the artists and craftsmen of many cultures. In fact, they are all represented in the (islamic) ornamental art at the Moorish palace of Alhambra in Granada, Spain[1]. In his book *Symmetry*, the great mathematician Hermann Weyl writes the following words:

> One can hardly overestimate the depth of geometric imagination and inventiveness reflected in these patterns. Their construction is far from being mathematically trivial. The art of ornament contains in implicit form the oldest piece of higher mathematics known to us.[2]

1. See e.g. *Curiosidades Matematicas de la Alhambra*: http://alhambra.dif.um.es/Alhambra/Curiosidad.html#GruposC.
2. [(169)], p. 103.

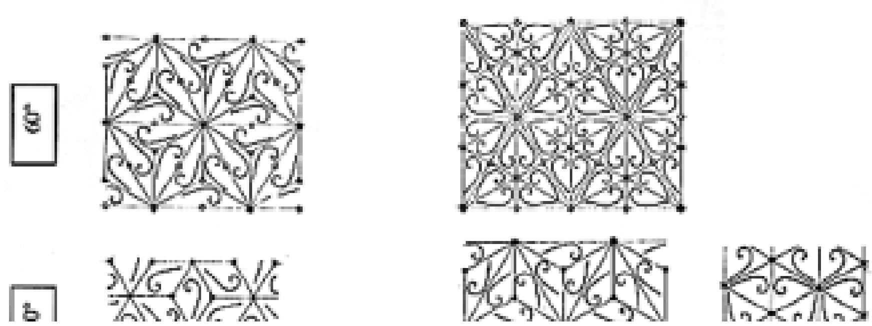

**Fig. 16. The seventeen different symmetry groups of wallpaper patterns**

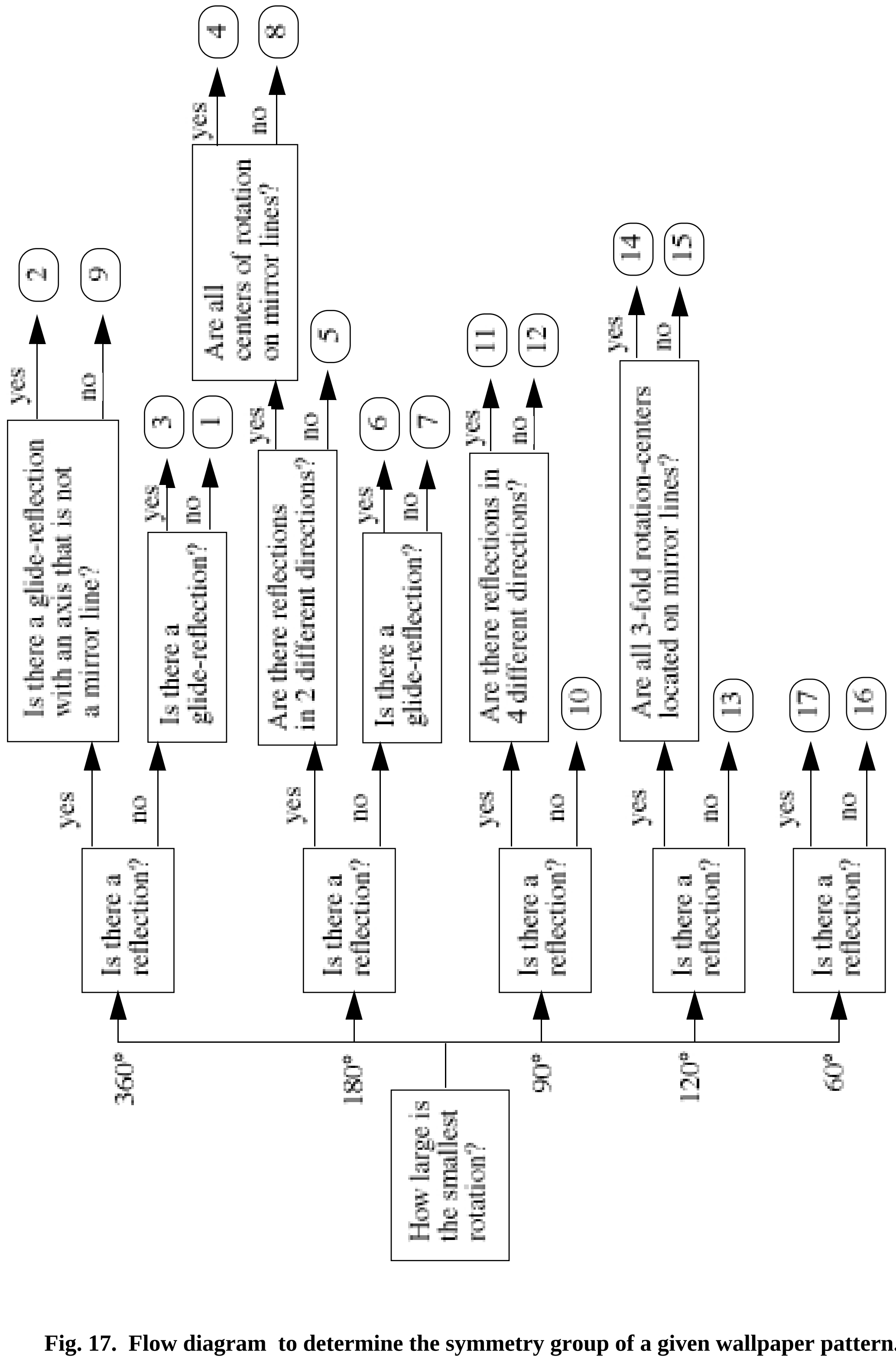


**Fig. 17. Flow diagram to determine the symmetry group of a given wallpaper pattern.**

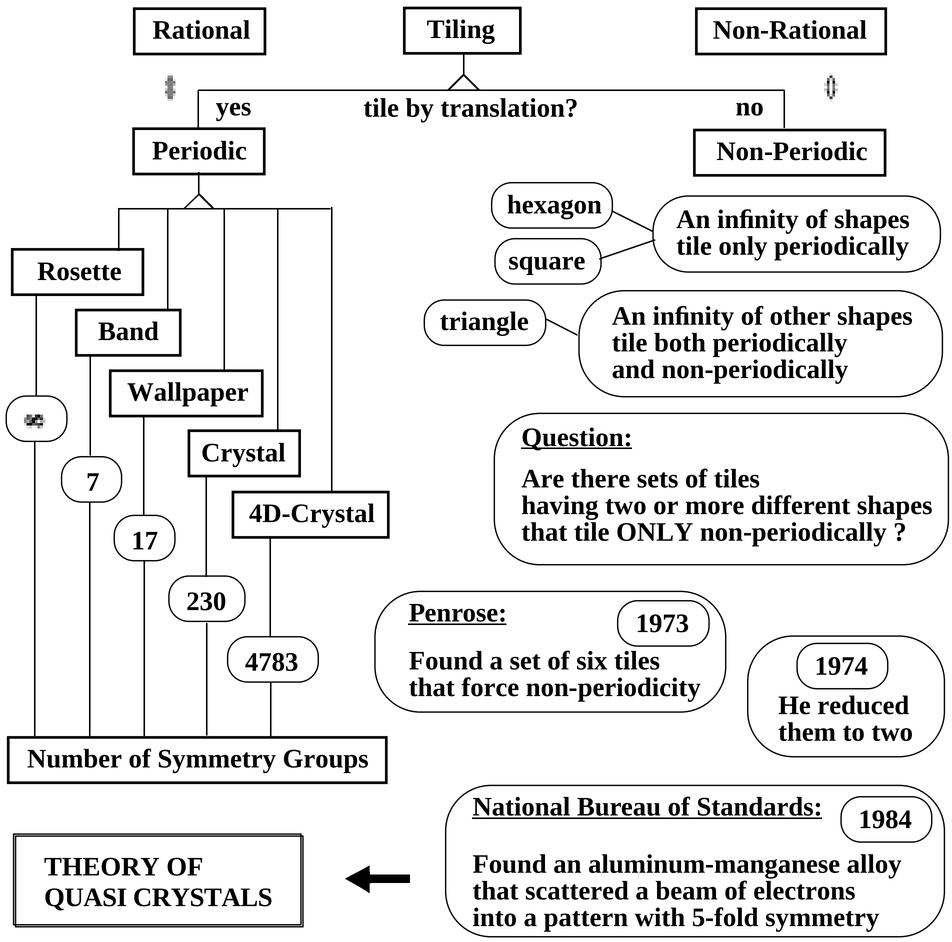


**Fig. 18. Crystals versus Quasi-Crystals: Periodic versus non-periodic tilings.**

# 8 Computer Supported Mathematics Education

## 8.1 The Computer as a Universal Mimicking Machine

In this paragraph I discuss a few general aspects of the computer and introduce some terminology that expands on the familiar concept of *virtual reality.* Then I define the concept of an *exiting exit* from a program and discuss its central role within the GOK project.

**Fact 13:** The computer can do *virtually* anything.

By this I mean that any kind of activity can (in principle) be simulated by computer. The computer could in this respect be likened to a *universal mimicking machine*.

**Definition 17:** *Virtual Reality* is a well-known aspect of computer-space. For the real, uncomputerized world I will reserve the term *Real Reality.*

Since a computer can realize (=implement) virtually anything, I also make the following

**Definition 18:** The term *Real Virtuality* will refer to any type of project that is realized by computer. *Virtual Virtuality* will denote a computer project that is *specified* but not yet *implemented*.

In these terms, the GOK - just like any other software design project - is concerned with the transformation from *virtual* to *real virtuality*. However, it is an important design goal of the GOK to help its users to connect with *real reality* in various ways. Using the computer as a tool in order to find out something that is so interesting that you want to turn off the computer and start exploring the discoveries within their own media. This could mean anything from ‘following a program link into the world of books - which is more profoundly explored off-line - to coming across some raytraced images of reflections in the program, and being shown where to find the corresponding mirrors, which are more directly explored by looking for yourself in real reality.

In fact, one of the main aims of the GOK is to aid the user in finding various forms of such *exiting exits’*, that imply leaving the computer to explore reality in other ways [Figure (19)].

**Definition 19:** An *exiting exit* is a transformation from *real virtuality* (=computer) space into some kind of real experience space (= *reality*) which is interesting enough for the experiencer to ‘forget what (s)he was doing’ for a significant amount of time, in order to pursue the reality of the new experience on its own. The *virtually exiting exits* are those, whose activities are still goverened by the computer, whereas the *really exiting exits* are the ones that involve forgetting about the computer altogether, in order to explore the newly discovered structure within its own medium.

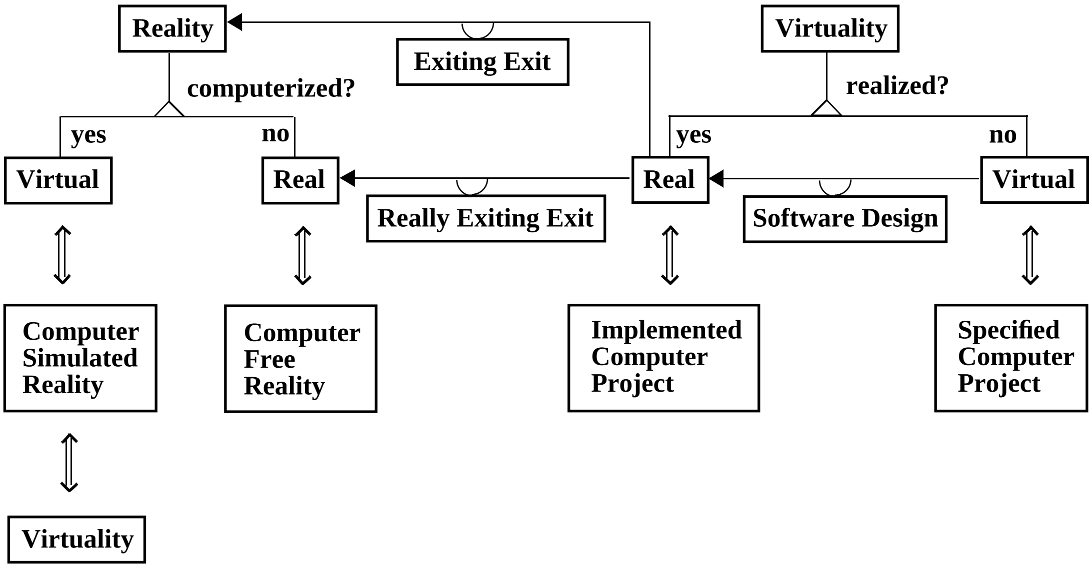


**Fig. 19. Exiting and Really Exiting Exits from Real Virtuality**

To emphasize what I mean by a *really exiting exit*, I return to the earlier example from optics. Due to the technological advancements of the plastics industry, there exist such things as high-quality flexible mirrors. Playing with such a mirror, flexing it in front of your eyes while looking 'through it', creates a world of strangely curved magical images. A *really exiting exit* would be e.g. to come across some examples of such images in the program - together with a link to where the corresponding mirrors can be found - and then become interested enough to go there, pick up some plastic mirrors and start flexing and looking for yourself.

As another example, consider the 'handling of clay' in a sculpting process. This would be *virtually exiting* if the computer told you how to proceed, and what materials to touch or not, but *really exiting* if you were free to explore the materials on your own, with no hampering restrictions.

## 8.2 The Educational Mathematics Programming Projects

As I mentioned above, for the last decade I have been involved with various software engineering projects within the field of mathematics education. I have 'suggested projects', as the term goes, by specifying ideas for mathematical software tools that would be nice to have around and resonable to implement. Below I have given a short list of these projects with some rudimentary descriptions. Most of these programs are still around, but some - notably *MapCon* , *Drawboard* and *MacDrawboard* - are history, and run only on extinct machines or under outdated operating systems.

### 8.2.1 MapCon

*Mapcon* is an interactive tool for studying *conformal mappings*.. The program allowed the drawing of a curve in one window, and its immediate mapping into another window according to a specified formula of type w = f(z). The program was written in Interlisp and ran on a Xerox-1108 machine, which is retired by now. A more detailed description is given in

Carlsson, C. & Gustafsson, J. & Heimdal, M. & Lantz, J. & Lundeborg, M. & Mattsson S., *MapCon , ett System för Laborativ Matematik* - ett projekt utfört förPMI-kursen av D-teknologer i årskurs 3 under ledning av Ambjörn Naeve på NADA & Matematik, KTH, 1986.

#### 8.2.2 MapAnalyze

*MapAnalyze* is a similar type of tool designed to display the effects of both linear- and non-linear types of mathematical transformations to various forms of geometrical objects. It was written in Scheme for the Macintosh. More details are given in

Ekman, T. & Kuna, R. & Ångqvist, T. & Scocco, W. & Sandbom, J & Vivas, J. L., *Mapanalyze* - ett projekt utfört för IPM-kursen av matematikerlinjens årskurs 3 under ledning av Ambjörn Naeve på NADA & Matematik, KTH, 1989.

#### 8.2.3 MacWallpaper

*MacWallpaper* is a program for the interactive study of wallpaper symmetries, i.e repetitive patterns of the wallpaper type. This is the program that formed the basis for the experimental part of the GOK symmetry project. I have used it a lot in my daughter's class as a mathematical 'stimulant', ever since she started out in first class in 1991. For me this was the natural beginning of my First Class Mathematics project. More about that below. The MacWallpaper program is documented in

Engström, A. & Koistinen, J. & Avatari, A. & Persson, K. & Grape, P. & Ottosson, A. & Andersson, Å. & Andersson, A., *MacTapet - användarhandledning*, projektarbete inom kursen Interaktiva System för Matematikerlinjens datalogigren, Stockholms Universitet. Projektledning och matematiskt grundarbete: Ambjörn Naeve, NADA & Matematik, KTH, 1988.

#### 8.2.4 Drawboard - Projektiv Geometry in Practice

*Drawboard* was an early version - in the Xerox Interlisp environment - of the program *MacDrawboard*, which was developed for the Macintosh a year later. A brief description is given in connection with MacDrawboard in the next paragraph. More details can be found in:

*Drawboard - Projektiv Geometri i Praktiken*, projektuppgift av elever vid Matematikerlinjens datalogigren, Stockholms Universitet under ledning av Ambjörn Naeve på NADA & Matematik, KTH, 1987.

#### 8.2.5 MacDrawboard - an Application for Applied Projective Geometry

The two programs -*Drawboard* and *MacDrawboard* - were both designed for the interactive study of plane *projective geometry*, which is the 2-dimensional geometry of points and lines and their various incidense-relations. They made it possible to *experience* (= *interactively experiment with*) some of the 'classical gems' of geometry - like e.g. the famous theorems of *Pappus.* and *Pascal*. This greatly increases the feel for what is going on. We can see directly what is happening in the theorem, without having to verbalize (= linguistically denote) each logical detail of the corresponding geometric configuration. The documentation of MacDrawboard is given in:

Andersson, N. & Berg, M. & Danielson, M. & Gyllensporre, D. & Johnson. C. & Järvklo, Å. & Levitte, T. & Paltzer, D., *MacDrawboard - en applikation för tillämpad projektiv geometri*, projektuppgift för D-teknologer i PMI-kursen DB330 under ledning av Ambjörn Naeve på NADA & Matematik, KTH, 1988.

#### 8.2.6 MacFlow - a Graphical Programming Environment

*MacFlow* is a graphical programming environment designed mainly for educational purposes. Its main purpose is to convey a feel for what an algoritm ‘actually is’, thereby contributing towards an increased understanding of what a computer ‘actually does’. Within the program, the user creates flow-charts with interconnected operators and variables. The interconnections are either of type data-flow or control-flow. The program supports the user in his efforts to conceptualize an algoritmic process, by visually representing both types of flow in the same diagram. This is a powerful mix which creates a ‘totality of presence’ of a computational process.

The algorithms can then be run in their entirety, like an ordinary program - or executed step by step under the control of the user. In the latter case, the current contents of the internal variables (intermediate data stores) as well as the corresponding state of control is clearly indicated and updated on the screen, which helps the student to obtain a mental picture of the ‘gestalt’ of an algorithmic process. More details can be found in:

Bång, A. & Hulthén, M. & Lindberger, P. & Nedlich, K. & Persson, J. & Sundberg, J. & Svensson, C., *MacFlow - a graphical programming environment*, PMI-project assignment in course DB330. Idea and Supervision: Johan Appelgren & Ambjörn Naeve, NADA / Matematik, KTH, 1989.

#### 8.2.7 PrimeTime

*PrimeTime* is a program that capitalizes on the ‘gaming-frenzy’ of our times and makes use of it in order to practice the basic arithmetical algorithms. The idea is to protect yourself when you are being bombarded with pairs of numbers. These numbers can be ‘neutralized’ by launching a missile containing their *sum* if you are playing in *addition mode*, their *product* if you are playing in *multiplication mode*, etc. The program is described more fully in:

Carlsson, P & Hartikainen, M. & Jörgensen, J., *PrimeTime - en användarhandledning för Macintosh*, projektarbete inom kursen Interaktiva System för Matematikerlinjens datalogigren, Stockholms Universitet. Projektledning och matematisk grundidé: Ambjörn Naeve, NADA & Matematik, KTH, 1992.

### 8.3 The First Class Mathematics project at the St.Erik Catholic School

In 1991 my daughter Ylva started in first class at the Sankt Erik’s Catholic School in Stockholm. Due to the kindness and interest of her successive teachers (Anette & Kristina), I have been allowed to work with the children and ‘talk mathematics with them’ for about an hour a week (and sometimes two). For the last six years, this has been my ongoing *First Class Mathematics* project. It has given me the opportunity to test some of my pedagogical fantasies ‘in real’ so to speak, and it has strengthened my conviction that *mathematics can be taught and learned in a first class fashion* from the start and all the way up through the educational system.

This is not the place to present the details of the mathematical smorgasbord that I have presented to Ylva’s classmates over these 6 years. It has included subjects such as

- summing the first 99 integers (1+2+3+ ... +97+98+99)
- figuring out how many different necklaces that could be made from 4 different beads
- figuring out in how many ways the class could line up in a row and appreciating the size of 25!
- playing with the faculty concept (by projecting Mathematica onto the overhead screen)

- building the Platonic solids, and exploring their duality in connection with Euler's formula
- exploring patterns by the help of MacWallpaper
- cutting up Moebius strips in various proportions, trying to figure out in advance what happens
- figuring out when a graph can be drawn without lifting the pen (leading to Euler graphs)
- doing arithmetic in different bases (how would we have counted if we had had 1 finger on each hand?)
- separating between the number and the figure (= its representation in a certain base),
- figuring out how many molecules of air that fit into an empty milk-brick ( Avogadro's number = $6.02 \times 10^{23}$),
- variables as boxes and solving equations as finding out what's in the box (showing how Mathematica does it)

In fact, I have found that many of the subjects that I have worked with are presented in an excellent little book called *Matte med Mening* (= *Meaningful Mathematics*) by Kristin Dahl[1], which I was not aware of at the time. Kristin presents a wealth of interesting material that can be presented to children at an early age in order to provoke and stimulate their interest and curiosity in mathematics. It's a shame that such a great book should "too expensive to buy in class" as one of Ylva's teachers so adequately put it.

Anyway, working with these kids has been great fun, and it has strengthened my conviction that the subject of mathematics can - and should - be presented to children in a much more constructive and thought-provoking way. Since the GOK project is heavily concerned with patterns, I will give a brief discussion of the pattern part of the project.

### 8.3.1 Working with MacWallpaper

This was the main activity during the first two years of the project. Using two Mac-II and a Powerbook 170, I assisted the children in working out their own patterns. Using an LCD-type of Overhead-screen projector, we were then able to project their different images up in front of everybody and hence to discuss them together.

The children were first encouraged to draw anything they wanted in the editor, and then play with the 17 different pattern-possibilities in which their basic image could be turned into a wallpaper. After modifying the drawing and choosing their favourite type of symmetry, the children were given paper printouts of the corresponding pattern and encouraged to expand on it further - by adding the dimensions of colour.

We also used an LCD-screen to project the patterns in front of everybody and discuss them together. In this way we analyzed their symmetry, and the children could practice how to discover the more subtle forms of symmetry - such as e.g. glide reflections. But at the same time we also discussed the holistic 'gestalt' of the patterns, i.e. the impression that they made on us as images. The idea was to convey an experience of the double-brained activity of mathematics - as described in Chapter (7.1). It was an amazing experience to feel the exitement of the children when they were confronted with these concepts. The discussions just did not seem to end.

1. See [(27)].

#### 8.3.2 Some General Observations

Of course, my smorgasbord of mathematics didn't manage to turn the oiltanker of learning by repetition. The reasons have to do with the general attitude towards early mathematics education, as presented in the discussion of Chapter (6.4). In other words:

**Opinion 12:** The fear of experimenting with new concepts in mathematics education is related to the algorithmic fundamentalism that still dominates the pedagogical thinking within this field.

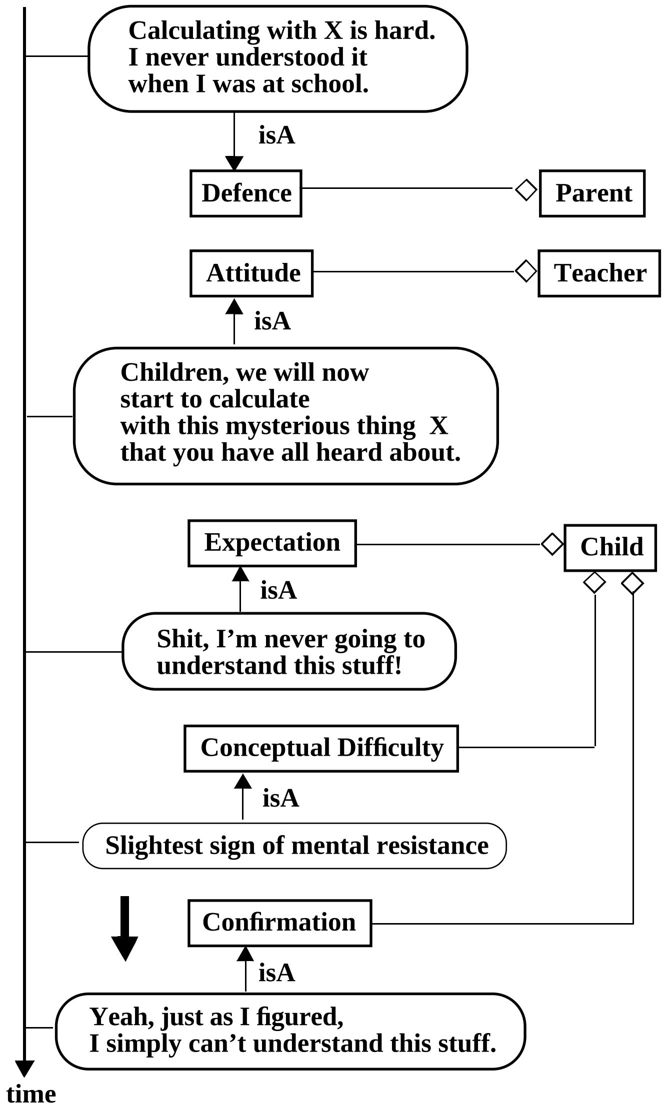


**Fig. 20. The X-anxiety pattern.**

The problem is not the learners but the teachers. To put it bluntly, *the children don't know that they don't know mathematics, but the teachers and the parents do*. And this attitude is very infectious: As an example, consider the following typical conversation - depicted in Figure

(20): The parent: “Calculating with *x* is hard. I never understood it when I was at school.” The teacher (later): “Now we are going to start calculating with that mysterious letter *x* that you have all heard about.” The learner reacts with something like: “Oh, shit, I’m never going to understand this stuff.” The learner (later - at the slightest sign of conceptual difficulties): “Yeah, just as I thought, I simply can’t understand this stuff.”

#### 8.3.3 A Variable as a Box with a Name and (maybe) a Content

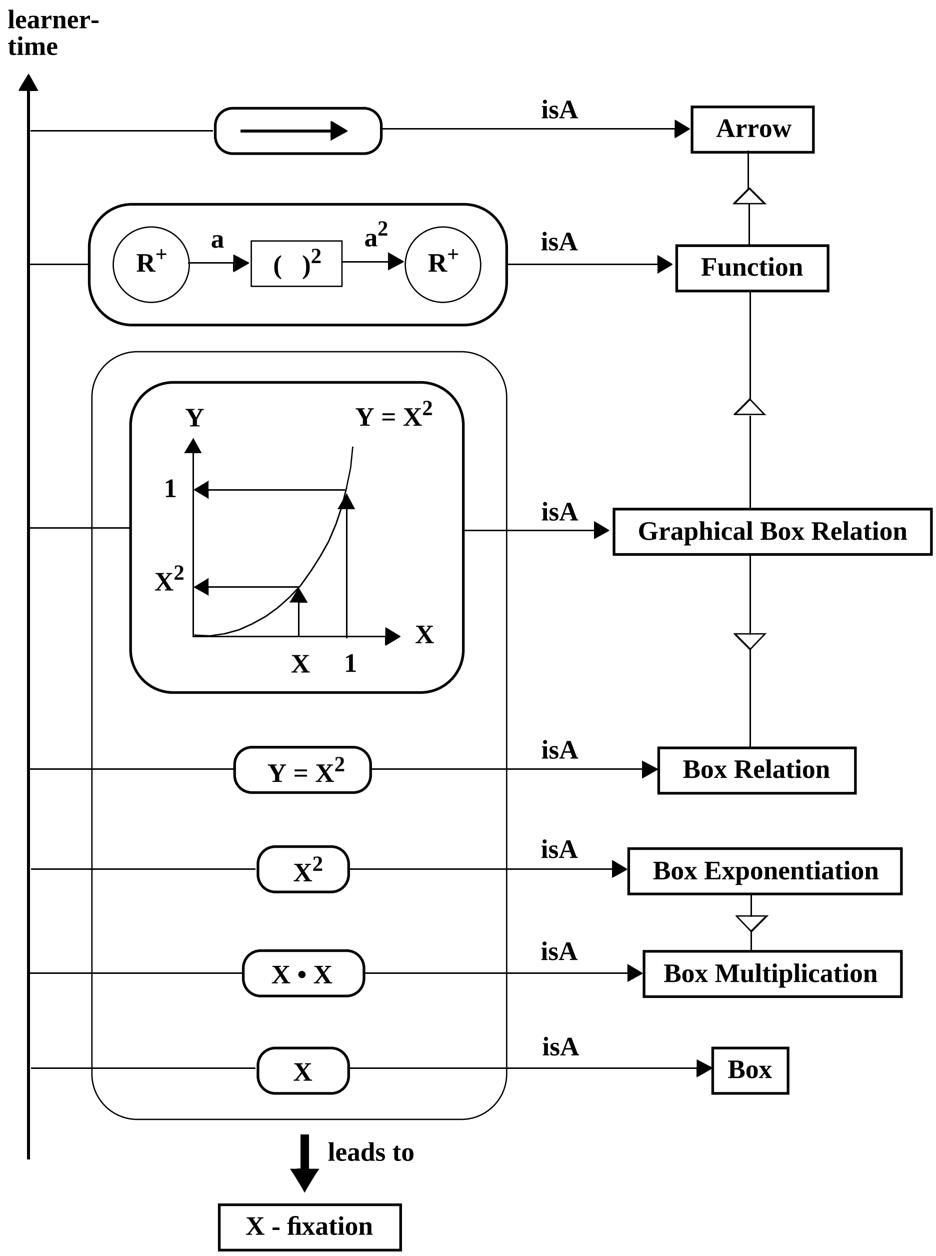


**Fig. 21. Building variable abstractions: Traditional approach versus ‘Box-calculus’.**

Teaching the children how to ‘compute with *x*’ should be handled in quite a different manner. from the way presented in Figure (20). I have used the analogy between a variable and a box as a starting point. Both have name and maybe a content - if we have put something inside the box - or equivalently - given some value to the variable. In this metaphor, an equation is a relationship between the contents of different boxes, and solving equations is equivalent to finding out what’s in the box.

This way of thinking about variables is very ‘concrete’ and easy to conceptualize for children. Moreover, it corresponds to how the program Mathematica™ treats a variable. If you type in the name of the variable, Mathematica responds with its value (= content) - if there has been something ‘assigned to it’ (= put in the box). Otherwise Mathematica responds with the name itself. By demonstrating this behaviour, and explaining it in terms of ‘computations with boxes’, I managed to communicate the idea of ‘what a variable is’ to each one of the 25 kids in my daughter’s class - at a time when they were about 10 year old.

Figure (21) shows some ways to think about *variables* and *functions* - two crucial concepts in gaining entrance to the higher levels of mathematics. The left part of this figure depicts different levels of abstraction encountered, and the right part shows these concepts mapped into the box analogy.

# 9 The Garden Of Knowledge - The Evolutionary Process

An important part of the Garden of Knowledge project is to document the interdisciplinary cooperative process. This is a notoriously difficult task - with which we have all struggled in various ways within the project group. In my capacity as initiator and overall architect of the Garden of Knowledge, I have had the role of formulating the basic conceptual structure, and in my capacity of mathematical gardner I have had the role of supplier of content (= knowledge source). It is from this perspective that I will try to describe the evolutionary process of the Garden of Knowledge project - as I have experienced it.

## 9.1 The Initial DIL Project

The DIL-project is concerned with *Design of Interactive Learning tools* - with computer support. My first contact with the project occurred in November 1995, when *Yngve Sundblad* invited me to a meeting (13/11) at The Royal College of Music (KMH). There I met *Kenneth Olusson* and *Bjarne Nyqvist* from KMH, *Bosse Westerlund* from Konstfack, *Peter Becker* from KTH and *Jorge de Sousa Pires* from Apple. We had a gengeral discussion on the nature of the project, and I ‘introduced myself’ and talked a bit about my background and interests within this field - such as the pattern generating program *MacWallpaper* and my educational project in *First Class Mathematics*. Then Kenneth and Bjarne presented a few of the multi-media based projects that were going on at KMH, and - as an external source of inspiration - we had a look at a CD-rom produced by Laurie Anderson.

At this time, the discussion that was going on within the DIL project was centered around the possibilities of using Jorge’s hypercard-based *Electronics Handbook* [1] as a starting point to test and develop different multi-medial forms of ‘pedagogical reinforcements’. I remember Jorge talking enthusiastically about the great possibilities of such ‘thinking tools’ - which was the term he used repeatedly. At our next meeting (in December) we chose an entry from the Electronics Handbook at random. It happend to be ‘modem’, and - as a form of home assignment until next time - we decided to think about various multi-medially supported descriptions of the concept of ‘modem’.

## 9.2 The Lecture

During the winter and spring of 1996, we eventually arrived at the conclusion that we should try to create a project in mathematics and one in music. At a meeting at the Royal Institute of Technology (KTH) in May 96 (15/5) I proposed that we should consider the possibilities to fuse these two projects into one - centered around illuminating the connections between mathematics and music. I was asked to motivate this further by giving a 2-hour lecture on the connections between the two subjects - as well as their historical evolution.

The lecture was given at the end of May (29/5). It started from the great ‘Greek dawn’ at Ionia, with the birth of the ‘rational project’ about 600 BC, featuring such great actors as *Thales*, *Anaximander* and *Anaximenes*, and later a ‘mystical rationalist’ named *Pythagoras* of Samos.

1. See [(35)].

“What is the basic stuff that the universe is made of?” - asked Thales - with the underlying assumption that the world can be understood from basic principles. This new belief in the reasoning power of the human brain was an electrifying spark that set the Greeks aside from earlier cultures. Soon everybody was reducing the world to basic building blocks - for Thales it was *water*, for Anaximander it was *air*, and for Anaximenes it was *pneuma* (the cosmic breath). The novely lay maybe not so much in the answers, as in the fact that the questions were being asked - not to the gods, like they had always been asked before - but to ‘dumb nature’ itself.

As a counterpart of these substance-oriented philosophers, Pythagoras claimed that the true nature of things are hidden in their *relations* - which gives them their form - and this nature can best be explored through the concept of *number* (= *figure*). This raised the philosophical battle between the substance-theorists (‘all is matter’) and the relation-theorists (‘all is form’) that has been going ever since - until it recently (≈ 1925) ended in a ‘quantum-mechanical draw’ expressed in the disturbing dual nature of light. The photon is neither particle nor wave, but sometimes the one, sometimes the other - depending on how we carry out the corresponding experiment. I devoted time to describe the unity (= holism) in the early quest for knowledge and discussed the Pythagorean form of interdisciplinary studies which involves connections between such diverse areas as astronomy, geometry, arithmetic, music, medicine and religion. [Figure (24)]. Within the field of music, I explained the fundamental discovery of Pythagoras concerning the connection between musical harmonies and the simplest rational numbers - such as the doubling (2/1) of the octave, the 3/2 of the quint, the 4/3 of the quart, etc. I also touched on the esoteric interpretation of the nature of numbers - such as the ‘male’ number 3 and the ‘female’ number 4 that unite in their common life-fruit 5(representing the child) - through the mediation of the so called ‘Pythagorean theorem’ ($3^2+4^2=5^2$).

I also followed a few threads of development forward in time, and described e.g. how the geometry of the Pythagorean quint-circle $(3/2)^n$ created the prerequisites for the syntheziser, built on the uniform quint, which was achieved in the eighteenth century (during the days of Bach) by introducing the irrational compromise $2^{1/12}$ for the frequency raise of a half tone step. I continued the story of the synthesizer and described how to play tunes with constant chords by making use of the so called Fourier Transform to calculate the pressure on each individual tangent. This gives an instructive analogy to the ‘material syntheziser’ used by quantum mechanics itself - where such wave-interference chords are played on the Fourier transform based tangents of the Shrödinger equation [Figure (22)] - in order to create matter in the form of wave-packets. I exemplified with the so called Heisenberg’s uncertainty relation - which describes the limits for the simultaneous accuracy that can be attained in measuring e.g. the position and the velocity of a particle (like an electron). With the Fourier transform, I showed how this amounts to the fact that waves of all frequencies have to cooperate in order to synthesize an infinitely sharp peak, i.e. a particle. In this context, the idea of the synthesizer as a form of time-machine - with possibilities to travel all the way from the sixth century BC of Pythagoras’ to the twentieth century of quantum mechanics.

I have a vivid memory of this lecture, since it is the only ony that I have so far held on this subject - the early evolution of western knowledge with an emphasis on the areas of mathematics and music. I also found the time to talk a bit about my educational project with First Class Mathematics [see chapter (7)], where I have experimented with trying to convey some carefully selected mathematical insights to the pupils of my daughter Ylva’s class. As I have described above, this project is based on viewing mathematical truths as a sort of logically tested fantasies

- where the right brain is fantasizing and the left brain does the logical testing. Only the fantasies that survive the logical tests are elevated to the status of mathematical truths. To encourage such a view of mathematics it is vital that the children are confronted with interesting structures at an early age - structures that can encourage them to foster their own mathematical fantasies. An example of such rich and interesting structures is provided by symmetrical patterns, such as rosettes, bands, and wallpapers. In the lecture, I showed examples of the experimental exploration of some wallpaper-patterns that was carried out in this spirit by a group of 9-year olds - using the program MacWallpaper - within the First Class Mathematics project. It felt like I managed to convey some of the mathemagical power of the structural exploration work of the kids!

Indeed, it was a very special lecture. Five of the six persons present were Rikard, Katarina, Klara, Kristina and Mattias - who all later became co-workers in the GOK-project in various ways. I interpret this as a sign that I managed to infect them with a substantial part of my enthusiasm for this holistically inspired knowledge project.

In the Figures (22) - (24) below, I have collected some of the patterns that I used in discussing these ideas.

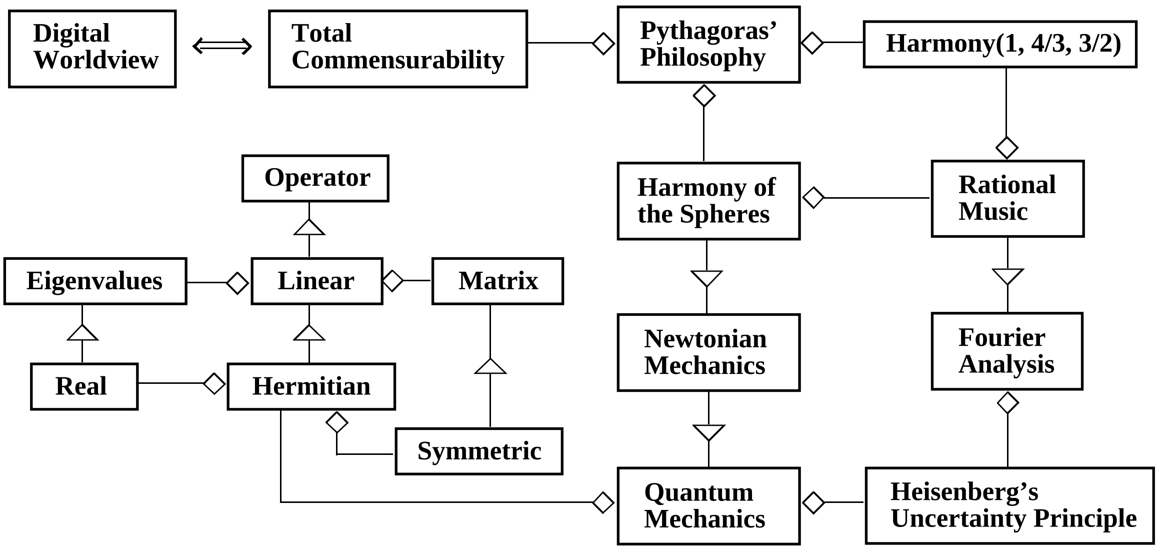


**Fig. 22. The evolution of the harmony concept - from Pythagoras to Quantum Mechanics**

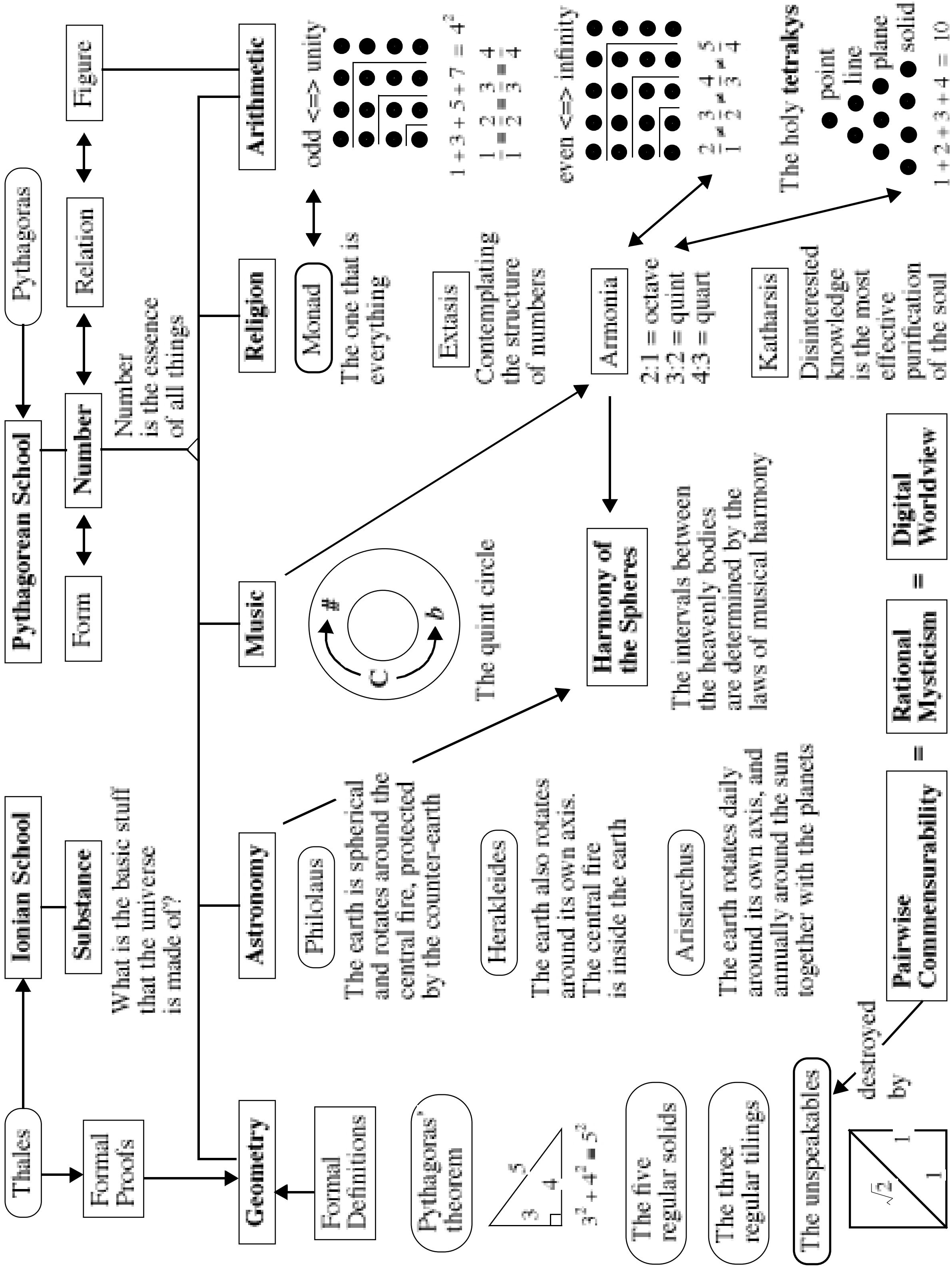


**Fig. 23. Overview of Pythagorean Philosophy**

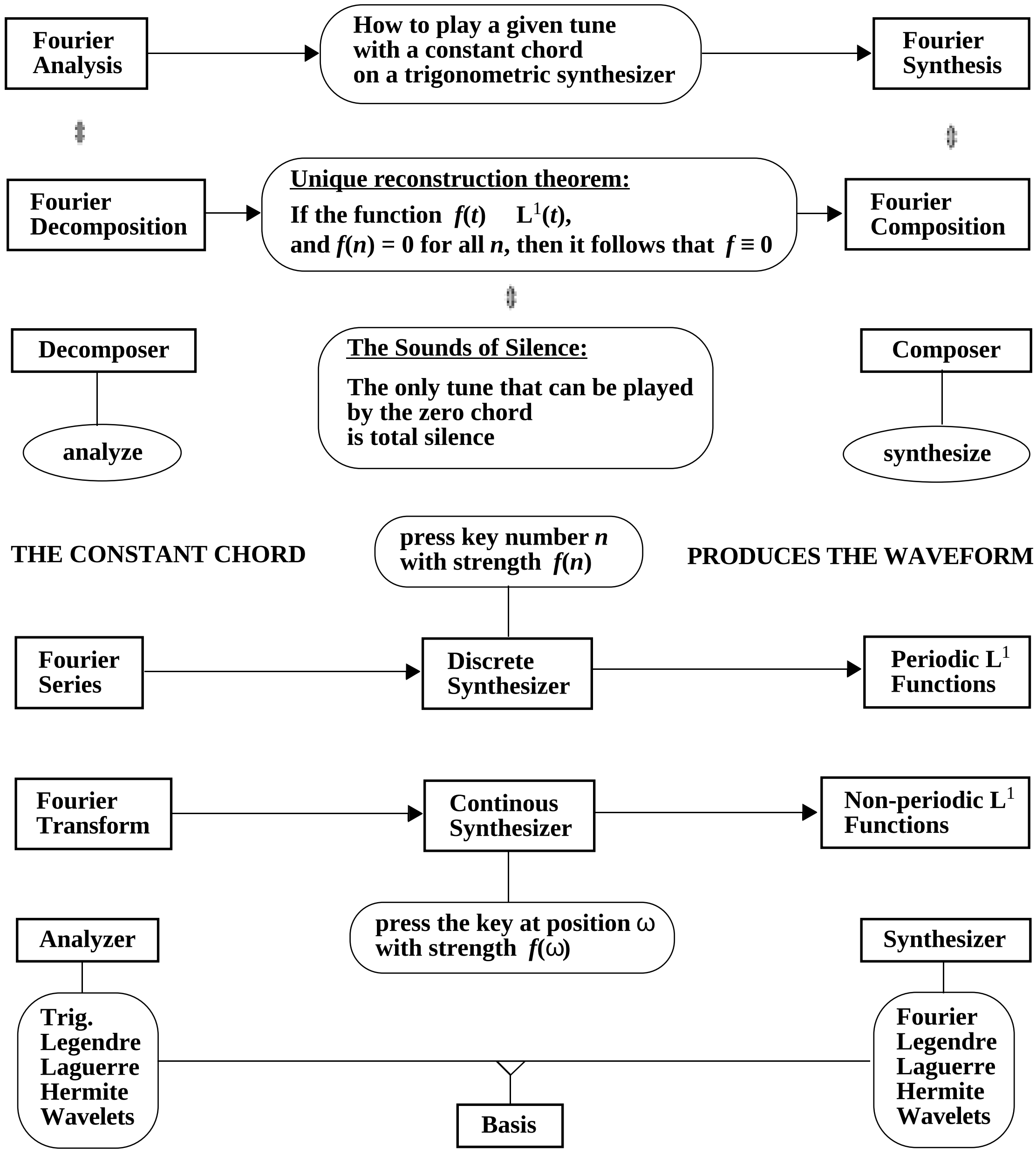


**Fig. 24. Playing a tune with a constant chord.**

$$\hat{f}(n) = (2\pi)^{-m} \int_{T^m} f(t) e^{-int} dt \tag{1}$$

$$f(t) = \sum_{Z^m} \hat{f}(n) e^{int} \tag{2}$$

$$\hat{f}(\omega) = (2\pi)^{-m} \int_{R^m} f(t) e^{-i\omega t} dt \tag{3}$$

$$f(t) = (2\pi)^{-m} \int_{R^m} \hat{f}(\omega) e^{i\omega t} d\omega \tag{4}$$

The synthesizer has a ‘change-basis-functions- type of menu with choice values like ‘Trig, Legendre, Laguerre, Hermite, Wavelets, ....

Coordinates of the vector $x$ in the basis $B$:

$$\hat{x} = [x]_B = [x^1, \cdots, x^n]_{\{b_1, \quad, b_n\}} \tag{5}$$

The vector $x$ synthezised back from its coordinates as a sum:

$$x = (\hat{x})_B = \sum_{i=1}^{n} x^i b_i \tag{6}$$

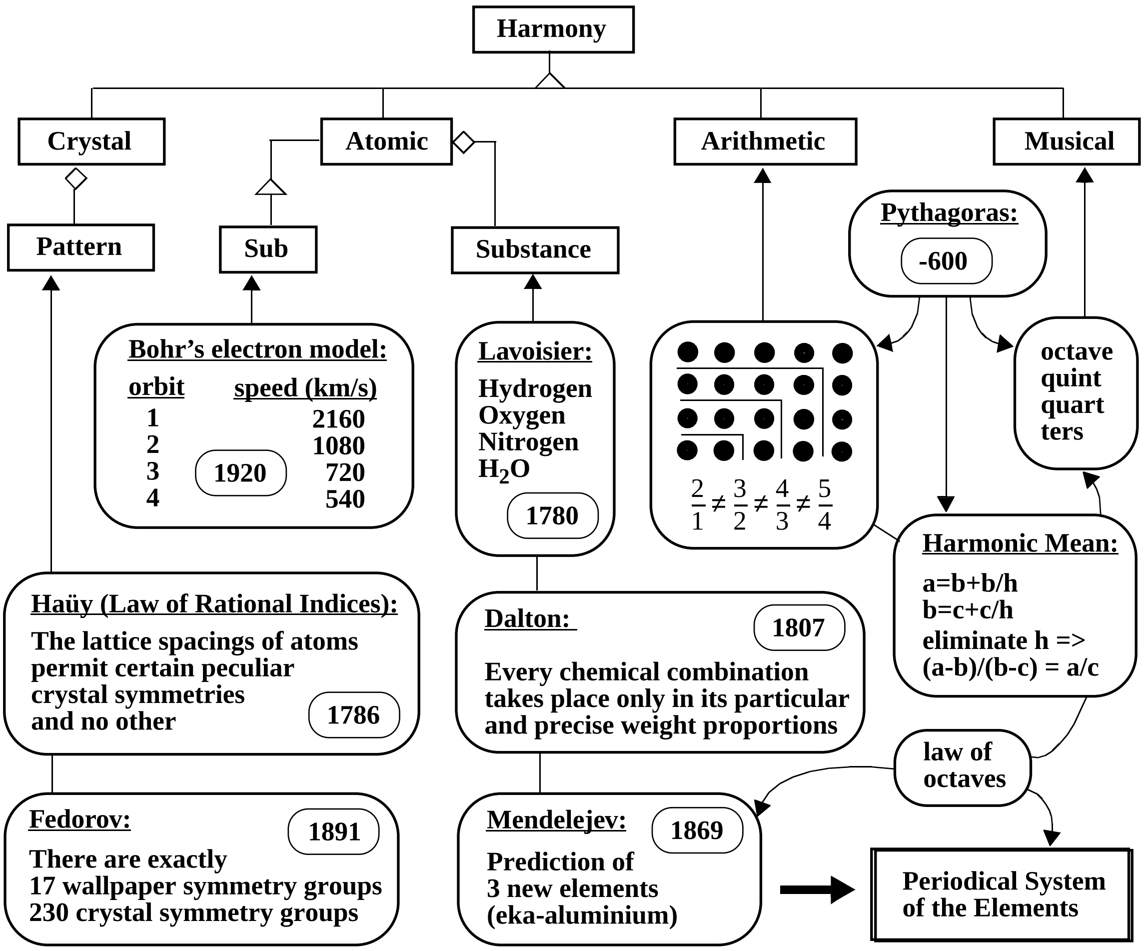


**Fig. 25. The evolution of pythagorean harmony.**

## 9.3 The Concept

Out of this lecture, the concept of a ‘garden of knowledge’ was born. Katarina was the one that introduced the metaphor of the garden. She imagined the garden of knowledge as a kind of computer game - where one would start out in an old and neglected garden and try to restore it to its past glory of knowledge and wisdom - with the help of the all-wise mystic and mathematician Pythagoras.

I have had many different working names for this kind of project before. *Mathematical Planetarium, Information-Art-Show, Brain-Train* and *Knowledge-Smogasbord* - to name a few. But the concept of a ‘garden of knowledge’ gave the metaphors a number of new and exiting dimensions. The teacher as a *gardener*, i.e. as a *cultivator* and a *pruner* of the knowledge which is gradually evolving out of the ever more complicated mosaik of information - this metaphor was coherent with the idea of the teacher as an *exformator* - i.e. the teacher as a support for the focusing on the relevant structure and the disregarding (= weeding out) of all the rest of the information.[1]

In short, a garden of knowledge seemed to be the natural environment in which to serve up an interesting smorgasbord of knowledge, adapted to the inevitable change - away from duty-oriented and towards interest-oriented learning - that is beginning to permeate the educational systems of today. To individualize the educational process by adapting the various corriculae to the individual - instead of the other way around - is a practiced both within academia as well as within ‘commercial education’ (in the form of tailored courses for employees).

## 9.4 The Big Think

The concept of a garden for handling knowledge is a very powerful idea-virus. It had a strong impact on my thinking - especially since it connected remarkably well with several of my old ‘favourite ideas’, as for example the concept of post-intellectualism. I have often argued that the so called ‘post-industrial society’ is better described as being ‘auto-industrial’ but ‘post-intellectual’. In the historic year of 1968, the Swedish State Bank instigated the so called ‘Nobel prize’ of economics[2] - thereby elevating the practice of economics into the ‘noble status’ of science. To me, this act symbolizes our definitive transition into the present post-intellectual state.

The trend of mathematics education that I have witnessed [see Figure (2)] - more and more of ‘know-how’ and less and less of ‘know-why’ - is a natural consequence of this paradigm shift. It now occurred to me that a Garden of Knowledge would be the natural step forward - into the *compost-intellectual* information society. Just as any garden, this one would contain an ‘intellectual compost’ - in order to break down outmoded thoughts and concepts and encourage their organic recombination in the form of new ideas.

1. In his book *Märk Världen* [(120)], Tor Nørretranders describes this process as ‘creating *exformation*’. Such exformation procedures form the basis of all types of learning, and are necessary in order to transform information into knowledge.
2. Riksbankspriset till minne av Alfred Nobel (The State Bank prize in honour of Alfred Nobel), to be awarded together with the real Nobel prizes.

:Since it is one of the aims of the GOK-program to support the concept-formation process and render it more visible in various ways, it became natural for me to refer to this program as a compost-intellectual philosophy s(t)imulator. The attempts to model the theorizing process in itself are based on an epistemology of science that has been put forward by Nicholas Georgescu-Roegen, who is one of the deep thinkers of the twentieth century. In his book*The Entropy Law and the Economic Process* he gives the following description of theoretical science: [1]

> Theoretical science is a living organism precisely because it emerged from an amorphous structure - the taxonomic science - just as life emerged from inert matter. Further, as life did not appear everywhere there was matter, so theoretical science did not grow wherever taxonomic science existed: its genesis was a historical accident. The analogy extends still further. Recalling that "science is what scientists do", we can regard theoretical science as a purposive mechanism that reproduces, grows and preserves itself. It reproduces itself because any 'forgotten' proposition can be rediscovered by ratiocination from the logical foundation. It grows because from the same foundation new propositions are continously derived, many of which are found factually true. It also preserves its essence because when destructive contradiction invades its body a series of factors is automatically set in motion to get rid of the intruder.
> To sum up: *Anatomically*, theoretical science is logically ordered knowledge. A mere catalog of facts, as we say nowadays, is no more science than the materials in a lumber yard are a house. *Physiologically*, it is a continous secretion of experimental suggestions which are tested and organically integrated into the science's anatomy. In other words, theoretical science continously creates new facts from old facts, but its growths is organic, not accretionary. Its anabolism is an extremely complex process which at times may even alter the anatomic structure. We call this process 'explanation' even when we cry out ""science does not explain anything". *Teleologically*, theoretical science is an organism in search of new knowledge.
>
> Some claim that the purpose of science is prediction. This is the practical man's viewpoint even when it is endorsed by such scholars as Benetto Croce or Frank Knight. Neo-Machians go even further. Just as Mach focused his attention on economy of thought without regard for the special role of logical order, they claim that practical success is all that counts; understanding is irrelevant. No doubt, if science had no utility for the practical man, who acts on the basis of predictions, scientists would now be playing their little game only in private clubs, like the chess enthusiasts. However, even though prediction is the touchstone of scientific knowledge - "in practice man must prove the truth", as Marx said - the purpose of science in general is not prediction, but knowledge for its own sake. Beginning with Pythagoras' school, science ceased to serve exclusively the needs of business and has remained always ahead of these. The practical man may find it hard to imagine that what animates science is a delight of the analytical habit and idle curiosity; hence, he might never realize what is the source of his greatest fortune. The only thing that exites a true scholar is the delight in adding a few bars to an unfinished symphony or, if he happens to believe in the ontological order of nature, in uncovering another articulation of that order. His interest in a problem vanishes completely the very moment he has solved it.
>
> Others say that science is experimenting. As far as theoretical science at least is concerned, this view confuses the whole organism with one of its physiological functions. Those who commit this error usually proclaim that "Bacon is science's John the Baptist". Naturally, they also blame Aristotle's philosophy of knowledge with its emphasis on Logic for the marasmus of science until Francis Bacon's time. Facts have never been more ignored. To begin with, Aristotle never denied the importance of experience; one eloquent quotation will suffice: "If at any future time new facts are ascertained, then credence must be given rather to observation than to theories and to theories only if what they affirm agrees with the observed facts"[2]. In relation to the time in which he lived he was one of the greatest experimenters and keenest observers. As Darwin judged, Linneaus and Cuvier are "mere schoolboys to old Aristotle". His teachings should not be blamed for what Scholasticism did with them. Finally, mechanics was already moving fast on Aristotelian theoretical tracks at the time Bacon's works appeared. Without the analytical habit which had been kept alive by Euclid's Elements and Aristotle's writings, Kepler, Galileo and Newton, as well as all the great men of sci-

1. [(57)], p.36.
2. Aristotle, *De Generatione Animalium*, $760^b$ 30-33.

ence that came later, would have had to join the Sino-Indians in contemplative and casual observation of nature. To the extent to which we may turn history around in thought, we may reason that without the peculiar love the Greeks had for Understanding, our knowledge would not by far have reached its present level; nor would modern civilization be what it is today. For better or for worse, we have not yet discovered one single problem of Understanding that the Greek philosophers did not formulate.

Roundabout this time I happened to think of the book *Tankar från Roten* [1] by Tage Danielsson - and combined with traditional UNIX jargon this led to the idea of the *root-priviledges of the gardener.* Another fertile analogy was the comparison to the biblical approach. Here we find a *Tree of Knowledge* growing in *Paradise*, which could be described as the *Garden of Innocence* (≈ ignorance). It now occurred to me that we were dealing with a ‘dual’ type of situation - a Garden of Knowledge, which by reasons of symmetry ought to contain a *Tree of Ignorance*.

In my mind, this tree became associated with a state of randomness, where thoughts would be generated ‘at random’ by making use of different types of ‘verbal one-armed bandits’, which I had been wanting to construct for a long time. When you pull the lever - instead of a variety of fruits - you would get a well formed sentence with the individual words selected at random out of suitable word-class repositories of nouns, verbs, adjectives, pronouns, preposition, etc. etc.

Moreover, in the contrast between the stochastic and the deterministic state lies the tension between the ‘irrational dice magic’ on the one hand, and a world model based on cause and effect - which forms the basis for the entire rational (≈ deterministic) scientific project - on the other.  Here lies the conceptual basis for a description of the statistical (≈ stochastical) way of thinking that has been applied with such enormous success in modern times - within such diverse fields as e.g. *quantum mechanics* and *financial mathematics*.

Yet another strength of the GOK metaphor was that it could naturally accomodate the allegory of *knowledge that is being transmuted into understanding*. If the brain succeeds in the process of constructing a cocoon out of its mental reflections, then each ‘catepillar of knowledge’ is given a chance to develop its own eventual ‘wings of understanding’. The analogy between the ‘seeker of understanding’ and the catepillar that spins itself into its own cocoon - based on an inner conviction (≈ instinct) of a necessary qualitative change -  emphasizes the role of the teacher in assisting the students to *nurture, focus* and *reflect* on their evolving patterns of information (≈ knowledge).

To sum up, in the metaphor of a garden of knowledge I felt there was a strong conceptual environment for future development work, since this metaphor naturally represents the two-step process of *weeding* and *reflecting* which is necessary in order to transform *information* via *knowledge* into *understanding*.

$$\boxed{\textbf{Information}} \xrightarrow{\textbf{weeding}} \boxed{\textbf{Knowledge}} \xrightarrow{\textbf{reflecting}} \boxed{\textbf{Understanding}}$$

**Fig. 26.  The exformation process as a two-step transformation of weeding and reflecting.**

1. ≈ Thoughts from the Root

### 9.5 The Pythagorean Mutiny

By the middle of June (12/6) there was an important meeting at Apple in Kista. Bjarne, Bosse, Peter Becker, Jorge, Rikard, Katarina, Klara, Kristina were present. Yngve and Kenneth were prevented to obtain. Within the DIL-project group we had been discussing different possibilities - broadening the horizon of possibilities all the time - but now we had come to the point of having to take some hard decisions as to what we were really going to put our efforts into. The students and designers were enthusiastic about the possibility of being able to work within the GOK-project as I have outlined it above.

Following a rather long introduction by Jorge, I presented our idea of project ‘Garden Of Knowledge (GOK) - supported by my overview-map of Pythagorean philosophy and the birth of the rational project [Figure (23)]. I mentioned Katarina’s idea of the garden restoration game, and discussed the GOK concept as a way to illuminate the relations between phenomena and concepts in general. In my presentation I tried to paint the same kind of ‘holistic perspective’ on knowledge that hade managed to captivate the students a few weeks earlier.

My talk lasted for about 15 minutes, and afterwards I had the feeling that it came across as a bit too short and ‘disconnected’. It was clear that the project was encountering strong opposition from the part of Apple, i.e. Jorge. “It must not turn into some kind of multi-medial playground” - was one of his comments. The most positive reaction I got from Peter Becker, who was the only one that expressed his undivided support for the ‘history-of-ideas approach’ to the subject. The rest of the DIL group kept a rather low profile and had no really explicit opinions as to the relative merits of the proposed approach.

After the meeting the students were competely desillusioned. “Its not going to happen- they didn’t like the idea” was the general reaction. I had to devote a lot of energy into convincing them that it wasn’t as bad as it looked. “I think they’re going to let us have a go at it, you just wait and see” - was my optimistic attitude.

And indeed. In a meeting with Yngve a few days later, we ‘cleared up some misunderstandings’ and got an OK to go on working with our concept of the Pythagorean garden of knowledge. En emerging project group was formed, where Kenneth Olausson and I would function as idea-generators together with a group of students and graphical designers (Katarina, Rikard, Klara, Kristina)

I would propose an overall conceptual structure for the garden, and Kenneth and I would propose various forms of content from the fields of music respectively mathematics. Later I was given the title of ‘overall garden architect’, and Kenneth and I were appointed musical and mathematical gardeners - responsible for the content of the corresponding knowledge patch.

The rest of the group would function as a sort of ‘interest-filter’. Whatever seemed to be interesting and intelligible enough would be developed further. Moreover, each member of the team would have their own independent functions. Klara and Kristina were to work with the graphical design - as ‘estetical gardners’, and Rikard - as a ‘programming gardener’ - would take on the coding of the prototype sketch that we planned to put together with the use of Director®. J remember that Rikard - rather reluctantly - let himself be persuaded into this by Yngve, since he did not think that he had enough programming experience to take on a project of this kind. According to Yngve, however, Director was ‘no match to learn’ - and he referred to all the successful programming projects that had been carried out in this environment as a part of various forms of esthetic multi-media education at GI and Konstfack.

Katarina, who was very enthusiastic about the project as a whole - but found it more difficult to envisage her own role within it - was finally persuaded to act as the ‘coording gardener’ of the GOK-project.

It was further decided that we were to work during the summer, and towards the middle of September we would evaluate the project, and decide if - and in that case how - to proceed. I remember expressing some anxiety concerning the possibility of working together during the summer, but we arrived at the conclusion that it would work out if everybody gave the project some ‘reflective thought’ from their own respective angle of things - and then we would ‘calibrate’ our thoughts later.

### 9.6 The Consolidation

During the rest of June I devoted myself intensely to applying the garden metaphor to the educational process as a whole - thinking about how the corresponding concept-formation system ought to be structured.

The environment of the garden gave a natural background for the presentation of various *elements* and *phenomena* , as well as a possibility to supply a variety of different *tools* to experiment with these. From the start I imagined the elements divided into two classes - depending on whether they belonged to heaven or earth. Among the *‘heavenly elements’* were of course the *sun*, the *moon*, the *stars* and the *planets*, and among the *‘heavenly phenomena’* e.g. *appearance*, *motion* and *disappearance of light*. Among the *‘earthly elements’* were *air*, *water*, *solid* (soil), *fire* and *vacuum*, and among the *‘earthly phenomena’* were *light*, *sound*, *heat*, *electricity* and *magnetism* [Figure (27)].

In the garden I also envisaged a number of *experimental tools*, i.e. different types of computer programs, to support the interactive exploration of various *elements* and *phenomena*. There would e.g. be a *Lyre* and a *Drum* for performing musical experiments, and a set of *Canonballs* to build pythagorean pyramide-figures. There would also be a set of*Tiles* - to represent the exploration of gemetrical patterns in general. As a basis for this activity we already had the program MacWallpaper [see paragraph (8.2.3)]. As presented in Figure (27) the lyre was associated with the *analogue* part of the music (= its *tone*), while the drum represented the *digital* part (= its *beat*). The lyre would be connected to the pythagorean monochord, and futher - through time travel - to the Bach piano, and the modern synthesizer. The drum would work as an entrance to the ‘rythm-laboratory’ - where one could experiment with building various types of rythm instruments, e.g. with the help of the so called ‘Chinese remainder theorem’ - which is of fundamental importance within the field of fast computing (= ‘modular arithmetic’)[1].

Towards the end of June I had a meeting with Kenneth (27/6), where I presented him with these concepts. Since he had not been present at the Apple meeting, this was the first time that I had the opportunity to introduce him to the concept of a pythagorean garden of knowledge.

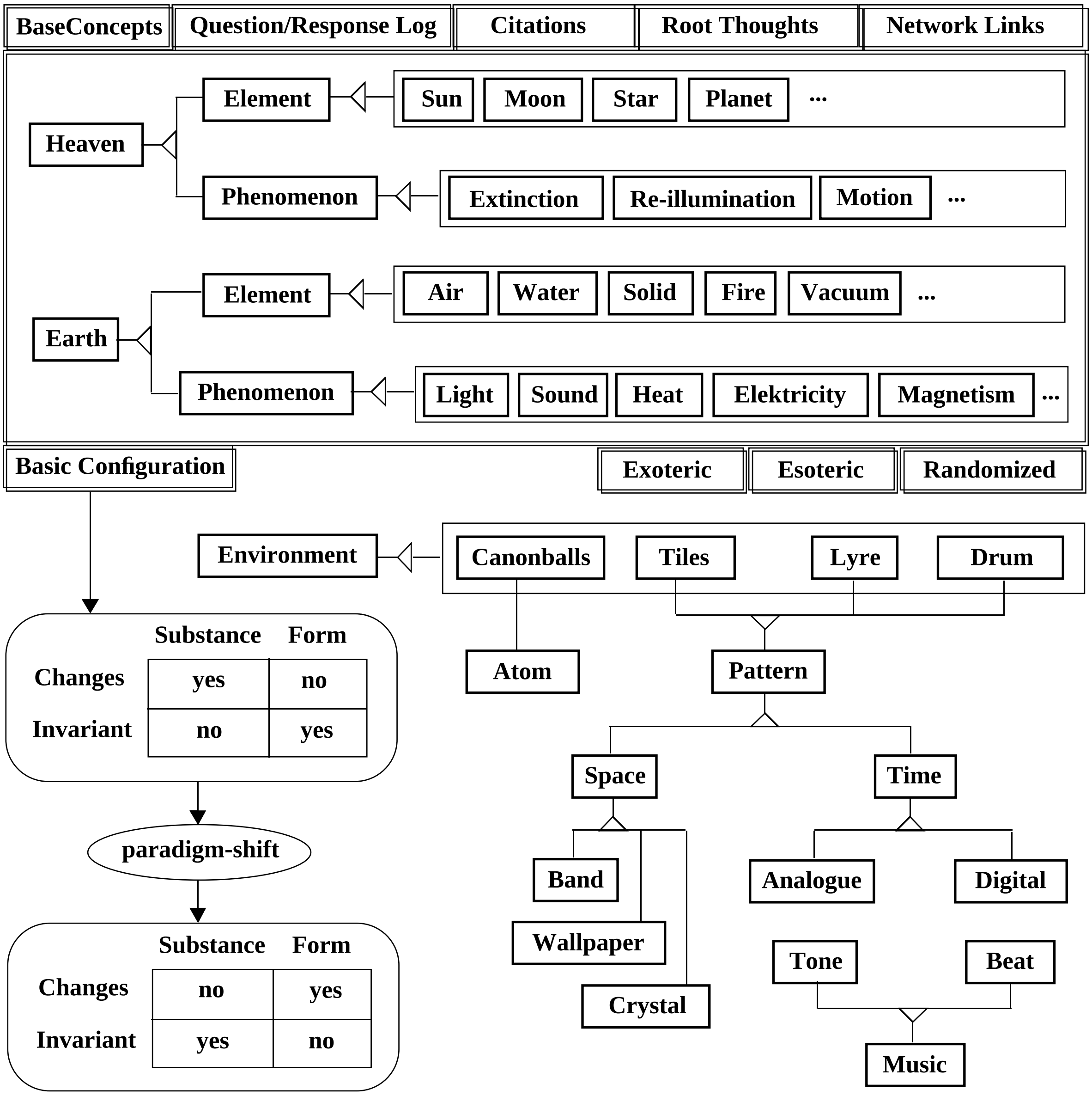


**Fig. 27. Heavenly and earthly elements and phenomena (97-06-27)**

Kenneth was instantly positive and very responsive to these ideas. He proposed that we should make use of a (brick) *wall* in stread of a drum - partly because it harmonized better with the

1. Se t.ex. Biggs, *Discrete Mathematics*, .....

concept of a neglected garden, and partly because we could extend it into a type of rythm tool that made use of bricks of different length in order to represent different time intervals. I liked this idea a lot, and we discussed many types of mathematical connections, as e.g. the representation of various number sequences by the corresponding rythmical processes.

It was a very constructive meeting. For the first time I felt some real enthusiasm for my thoughts about the GOK from somewhere else than the part of the students. We definitely had an interesting set of connections between mathematics and music to explore!

### 9.7 The Emergency Hack (Pythagoras of Crotona)

In July 96, I went to California for a month. I continued thinking about the structure of the GOK - as well as to gather more information. During the summer I realized that there should be a third mode of operation (= state) in the garden - apart from the rational and the random - namely the *esoteric* (≈ *spiritual*) mode [Figure (27)]. In this way the program would incorporate the fundamental difference between the *material* (≈ *exoteric*) and the *spiritual* (≈ *esoteric*) way to look at the world - thereby contributing towards bridging the devastating gap between science and humaniora. [Compare Figure (7)].

As a ‘humanist’ it is just as confining to try to conserve your ignorance of technological and scientific matters, as it is for an ‘scientist’ or an ‘engineer to try to ignore philosophy and history of ideas. The important point is that there should be ‘room enough’ in the GOK in order to allow the new generation of students to develop their own world-views in a non-biased way.

Back from California, I persued the esoteric trail. Manly P. Hall’s classical work [(64)] was my main source of reference. I collected text material and classified articles according to their relations to the elements and phenomena - as pictured in Figure (27). I consulted Koestler [(89)], Murchie [(104)], Heath [(68)] and several other ‘gardeners’ - well versed in the history of ideas.

I concentrated on three basic types of questions that could be asked about any combination of elements and phenomena - in exoteric as well as in esoteric mode. The basic, overview-question was the following: *‘What is its material/spiritual nature?’*. This fundamental question was then complemented with three different aspect-questions - namely, *‘What does it consist of?’*, *‘How does it behave?’* and *‘Why does it behave this way?’*. I spent quite some time with the question of the *nature of light* - which is one of the most subtle questions that physics can ask.

I wrote condensed descriptions with various *basic configurations*: light as *particles*, light as *waves*, light as *photons* ans light as *colours*. I also formulated the corresponding types of questions and responses within the fields of *heat*, *energy* and *work*. The aim of this activity was partly to lillustrate the nature of physics, but first of all to show by example how a collection of scientific insights might be organized in order to provoke student interest. In this activity, I started from the content and made use of the traditional text-medium - well aware of the fact that the ‘multi-medial dimensions’ would be added later. I wanted to have a rather large collection of informational raw-material to work with later, when we were going construct a first prototype.

Within the project group, we pretty much followed our own trails of thought during the summer. When we met up again in early August, we had some problems in coordinating these trails.

I envisaged a working prototype within a month (= middle of September) - a prototype that would have to include some kind of interactive structure on top of a carefully selected set of interesting facts - in accordance with the ideas discussed above. In view of the earlier criticism from the meeting at Apple, I felt that the informational contents that we were going to present in this prototype would have to be interesting enough to convince the sponsors of CID - within academia as well as industry - that the GOK project was something quite different from the ordinary ‘childish educational programs’ that seem to flood the market and contribute to giving the entire concept of multi-medial education a bad name.

During the summer, our graphical designers - Klara and Kristina - had worked out several interesting suggestions for shapes and symbols that would appear in the GOK. At this time we were looking for some kind of ‘icon-language’, a way to graphically represent the various elements and phenomena, whose combinations we wanted to describe and explore.

As time went by and August passed into September, it became increasingly clear to me that we were not going to have any prototype at all to show on evaluation-day, which was coming up on September 12. A day and a half before ‘demo deadline’, Rikard and I started up an ‘emergency hack’. During a 30 hours marathon session, we implemented my text-based design, which was built on top of a classical picture of Pythagoras in his garden of knowledge.[1] In the upper left corner, we had a yellow and and a red apple, that corresponded to the exoteric (= material) respectively the esoteric (= spiritual) state of the program. There were also a whole range of questions, comments and links concerning subjects such as *music*, *patterns*, *geometry*, *physics*, *astronomy*, and *philosophy*. The questions were ‘hardwired’ to certain elements and phenomena and could not be asked freely and in relation to any selected combination of elements and phenomena. The ‘answers’, i.e. the thoughts from the root, as well as the links (= citations from other gardeners) were mainly text-based with a few scanned images scattered here and there.

Maybe I can best describe the flavour of this ‘Pythagorean Prototype’ of the GOK, by translating an article by Titti Hasselrot, titled *Play it again Pythagoras*[2], where she describes a demo of the program which I gave her as CID on September 30, 1996.

### 9.8 Play it again Pythagoras

> Ambjörn Naeve invites us into his garden of knowledge, which means starting up his computer and the multi-medial learning environment that is emerging here at KTH in a close collaboration between engineers, scientists, musicologists, graphich designers and artists. Several of the university colleges around Valhallavägen are involved in this interdisciplinary development process - including The Royal Institute of Technology, The Royal University College of Music, Konstfack and the Dramatic Institute. Industrial interest includes Apple as well as the Swedish Institute for Systems Development. The basic resources are provided by CID (Center for User Oriented IT-Design), which is a recently formed competence center at KTH. The aim of the Garden Of Knowledge project is to provide a wealth of material for people that are curious about all sorts of ideas - with an emphasis on their cultural and historical development process. Not only ‘engineering concepts’ such as the wheel, the printing press or the electric light bulb, but also more philosophical and esoteric ideas, such as ‘god’, ‘love’, or ‘music’.

1. ‘Pythagoras of Crotona’, by Knapp [from *The Secret Teachings of All Ages* by Manly P. Hall, (64)].
2. *Notbladet*, nr 11, 1996.

### 9.8.1 The Esoteric Dimensions of Reality

The first appearance in the garden - i.e. on the screen - is the old greek ‘mathemagician’ Pythagoras, one of the founding fathers of science. Now you have to choose whether to click the yellow or the red apple up in the left hand corner. Ambjörn Naeve - the basic architect behind the project - explains: “If you click the yellow apple, the program transits to its material (= exoteric) state. Here you can browse through the historical development of ‘materialistic ideas’ - all the way from Pythagoras ( ≈ 600 BC) up to present times - and study questions of an extrovert (material) nature. If you click the red apple, the program transits to the esoteric (= spiritual) state, where you can study more introvert types of questions.”

Ambjörn Naeve is well aware of how provocative it can be to speak of esoteric matters within the scientific community. The important thing is that the new generation of students will be allowed to build their own world-views in a way that has not been biased from either the material or the spiritual side. In this respect the Garden of Knowledge project can be seen as an attempt to ‘breach the gulf’ betweeen science and humaniora. It is just as limiting for a ‘humanist’ to remain ignorant of scientific matters as it is for a ‘scientist’ to disregard philosophy and the history of ideas.

### 9.8.2 The Musical Harmony of the Spheres

So, let us click the lyre - which leans against a wall in Pythagoras’ garden - because the lyre is the entrance to the field of music. What did Pythagoras think about music - almost six hundred years before Christ? If we take the exoteric route into the material we get into Pythagoras’ theory of harmony. He was the first to quantify the natural harmonies of two vibrating string, and he showed that the harmonious tones corresponded to simple, integral ratios of the lengths of the strings. In this way Pythagoras established a profound connection between mathematics and music. “He was the first to mathematize music”, says Ambjörn Naeve, who is himself a mathematician with a strong musical interest.

If, in stead, we enter the lyre through the red apple, we encounter a beautiful engraving of the harmonious music of the spheres, which represents Pythagoras’ view of the world - and the interplay between its material and spiritual forces.

The GOK program is far from finished. What will we see if we click the red or the yellow apple in the future? What will we see that has to do with music in our own age? Is it really still possible - in this day and age of specialization - to envisage the connections between the philosophical or humanistic perspective on the one hand, and the scientific or engineering perspective on the other? “Yes”, says Ambjörn Naeve, “let me give you an example. On the one hand, we have in our days the art form of so called ‘minimalistic music’ . On the other hand, within science and mathematics we find many patterns of regularity being expressed with the help of computers. The relationship between minimalistic music and the theory of patterns is just as profoud as the relationship between Pythagoras’ theory of harmony and his philosophical world-view. It’s just a matter of seeing it.”

Being a geometer, Ambjörn Naeve takes a leap sideways and clicks into the field of Patterns. He demonstrates the 17 different repetitive principles that exist for constructing wallpapers in the mathematical sense. “For example, take a wallpaper with this repetitive effect. Add instructions concerning the instruments, the tempo and the volume that you want for the music. Then you can try out what this wallpaper ‘sounds like’!”

### 9.8.3 Transforming Knowledge into Understanding

We want to cultivate knowledge so that it is transformed into understanding. This means that we support a change of the traditional teaching role, away from ‘tenured preacher’ and ‘knowledge-filter’ towards the new role ‘gardener’ and ‘knowledge-cultivator’. “Students with some amount of extra curiosity is the primary target group for the Garden Of Knowledge program during the development work”, says Ambjörn Naeve, “but we all hope that the program will eventually dissipate down to younger age groups.” Through the GOK program, the user should also have access to various databases - as well as to books and other learning tools. “Book publishers cannot go on keeping their secrets”, he says, “a book can no longer be treated like a pound of butter!”

## 9.9 The Hyperbolic Symmetry of Escher's Angels and Devils

The demo took place on the inaguration day of the new CID localities - September 12, 1996. The emergency hack was well received by the audience, and - in my opinion - contributed decicively towards a continuation of the GOK project. It was, however, put together in a way that created tensions within the group. The graphical designers (Klara and Kristina) felt bad, because their part of the work didn't show up in the prototype, but was documented separately - on paper. This was unfortunate and affected the atmosphere within the group in a negative way - something for which I must take full responsibility. Seen in retrospect, I should have predicted this reaction, but I was simply too concentrated on producing a working prototype in time for the demo - a prototype which must have at least some measure of both function and content - to be aware of the negative energy that I was creating. Poor Rikard - whose heroic work had delivered the implementation 'just in time' - was caught in a psychological cross-fire, which made his work-situation considerably more difficult than it would have been otherwise.

Following the CID inauguration, it was decided that the project should be allowed to continue, and - in a meeting with Yngve - the next phase was organized. Bosse Westerlund was enlisted as project manager and esthetical gardener, which increased the esthetic competence within the group. This was important - especially since Klara quit the project at this point. It was decided that we would keep working during the fall, with a new demo in December - in order to evaluate the project and decide about further continuation.

During the fall Kenneth was taken ill, which grounded the music part of the project. Because of this unfortunate turn of events, the second prototype - which was presented on December 16 1996, did not contain any new musical material.

The critique that was voiced against the first prototype was basically founded on the embarrassing fact that the multi-medial content was blatantly missing. This fact - combined with the psychological tensions that had been created by the emergency hacking of the first prototype - led to a 'restart' in connection with the second phase of the GOK project during the fall of 96. Now we were to prioritize the multi-medial components, in order to explore what they could offer in terms of presentation support - in relation to the first, text-based prototype.

We decided within the group that the second prototype should be more 'linear' than the first - i.e. it would contain fewer possibilities for choice. In return for this restriction, it would - as a form of compensation - contain much more of multi-medial support. This decision, which may seem surprizing, was due to the inevitable programming difficulties that are connected with the creation of multi-medial interactivity. In spite of Rikard's great work, we were simply too understaffed on the programming side to be able to overcome these difficulties within the timeframe of the project.

We therefore made use of two well-known principles of effective design - namely *compromize* and *reuse*. I went through my old documentation videos, and came up with some material that I had been involved with producing about 7 years ago - in cooperation with Göran Adolfsson at UtbildningsRadion, Sveriges Telvision.[1] From this material - as well as from some of my own geometric animations from the old Symbolics days - I collected various pieces that had to do with geometry, and Rikard helped me to digitalize the material and turn it into Quicktime format. This collection then became our basic film repository for the second prototype.

One day, in a meeting with Katarina, I drew up a use-case scenario on the white-board by our usual group meeting place - the ‘kitchen roundtable’ at CID. The presented material was chosen both for its interest, as well as for its usefulness to demonstrate various forms of multi-medial support. Katarina took the whole thing down on paper, and our story-board became the basis for the implementation. We concentrated on geometry - presenting subjects such as e.g. *Pythagorean numbers, crystalline structures* and *hyperbolic geometry.* We used analogies from *optics* to explain the idea of a *geodesic curve,* and showed a film (and a computer animated sequence) to illustrate that the fastest route between two points is not always a straight line. We analyzed Escher’s famous woodcut ‘Circle Limit IV’ (Angels and Devils) to show that it represents a kind of wallpaper symmetry that cannot exist in the (ordinary) euclidean plane - since it contains both 3-fold and 4-fold centers of rotational symmetry.

The second prototype(the Angels & Devils) also contained sound.We had no sound recording facilities at CID, but through the courtesy of Osqradion at KTH, we were given access to their facilities. Magnus Skantz, Rikard Linde and myself were involved in this enterprise. Two days before the demo, we brought a powerbook to the sound studio. Then I read the text to each film ‘as it ran’, and Magnus taped it. Since the films were rather big (= took up a lot of disk space), we had to copy them one by one across the net.

On demo-day (Dec. 16, 1996) the Angels & Devils prototype was well received. Many people found the content interesting, although the degree of interactivity was not so high. Several people also commented on the differences between the first and the second prototype.

## 9.10 The Grand Epistemologic Scheme

After this demo it was decided that the GOK project would continue during the next year - at least until the end of June 97. By that time the entire work-plan for CID would be up for renewal anyway. Having produced one interactive, text-based prototype and then a (more or less) non-interactive multi-medial prototype, we were now eager to come up with a prototype that was both interactive and multi-medially supported.

Up til now, the monumental figure of the ‘knowledge-gardener’ Pythagoras had dominated our conception of what the garden would be like. As the idea of a more distributed form of of ‘gardenership’ took hold within the group, I came up with the concept of a *knowledge-patch*[1] to adapt the garden to this idea. Each knowledge-patch would have its own gardener, who would be subjectively responsible for the knowledge presented there. These patches would then be linked together into some form of *knowledge-patch-wor*k[2]. From this patchwork I later developed the idea of a *Knowledge Manifold* - see Chapter (11) - by connecting up with some old ideas about ‘calibration by coherence’ related to the concepts of *direct-* and *inverse limits* from *Category Theory,* which is a modern branch of mathematics [see Chapter (13)].

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1. The series was called ‘In the middle of Mathematics’ and it consisted of six 30-minute programs that illustrated different aspects of the subject. I was involved in making a program on non-euclidean geometry, together with Lennart Carlesson and Torsten Ekedahl. Hans Wallin, Hans Riesel and Claes Löfwall were involved with making other programs in the same series.

1. ‘’Kunskapstäppa’ in Swedish. This concept is connected to the idea of being ‘master of your own patch’ (= ‘’herre på täppan’)

2. ‘’Kunskapens Koloniträdgård’ in Swedish.

During the winter I made an attempt to formulate an overall epistemologic structure of the GOK - an attempt that is presented in Chapter (11) below. In a CID-seminar of April 23, we gave a presentation of our project, where I described the GOK program as a ‘compost-intellectual philosophy-s(t)imulator’. The meta-structural pattern behind this analogy is shown in Figure (30). The linguistically based concept formation (= ‘concept geometry’) of Chapter (11.2) forms the basis for the epistemological classification of Figure (38) based on successive ‘layers of disregards’. The topology of such *multi-layered equality* is presented in Chapter (12), and it forms the basis for effective design of the knowledge-components of Figure (37).

### 9.11 The Compromise

Within the project-group it became gradually clear that the navigatonal aspects of concept geometry presented too many problems to resolve within the 2-dimensional setting that we were working in. We therefore reached a sort of compromise, based on an idea that Bosse had picked up from a CD-rom presentation, and which allowed us to stay within the same presentation map, while browsing the contents on different levels. Together with the idea of a scrollable plane that wrapped around both horizontally and vertically, this formed the basis for the third prototype, the development of which is described in the group-report [(97)].

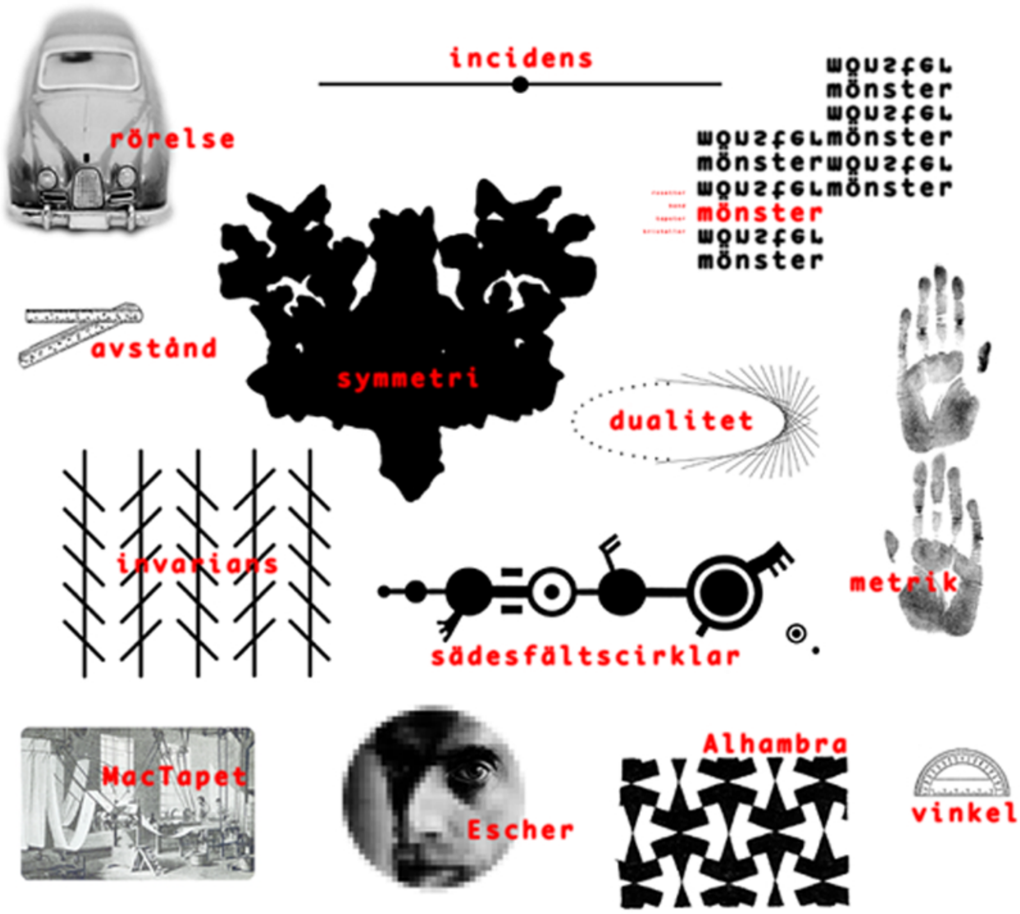


**Fig. 28. The geometry map of the third GOK protptype.**

# 10 The Subjective Observer

## 10.1 The Mental Control Room

The most fundamental of all human assumptions is the concept of ‘I’. This assumption creates a distinction between ‘me’ - ‘not me’, ‘inside’ - ‘outside’, ‘subjective’ - ‘objective’. This is a fundamental distinction which penetrates us deep enough to shape the structure of experience itself. In fact, it creates the basic framework that we use in order to explore and interpret reality. The process of ‘finding out what’s happening’ starts on the day we are born - if not earlier - and it goes on constantly throughout our lives. The external tools that are available to us in this quest for understanding are our senses, and the corresponding internal tool is our brain.

We can picture the brain as a person living in a closed room inside ourselves. This room has no windows and no doors, and the brain is forever shut up inside it. On one of the walls there are five monitors, one for each of our senses. There is also a row of levers, which the brain can pull in order to generate interaction with the outside.

The brain is constantly watching the monitors on the wall - studying the signals that they transmit, trying to figure out what goes on outside and how to interact with it. If we think of ourselves as space-ships - vehicles moving about in space - then the brain is the person in the control room, controlling the ship by watching the sense-monitors and pulling the action-levers. I prefer to think of the levers of action as the tangents of a large ‘action-piano, upon which the brain is able to play different tunes. Within this metaphor we could state

**Fact 14:** Life is playing your action-piano in harmony with your sense-music.

Of course, what is meant by ‘harmony’ (= ‘coherence’) is ultimately decided by the brain itself. We will have more to say about these concepts below.

## 10.2 Birth - The Landing on a Strange Planet

On the day we are born, our space-ship is indeed landing on a very strange planet. Our previous communication line, the umbilical chord, is completely cut off, and our sense-monitors are suddenly going bezerk - transmitting a lot of strange and unintelligible signals to the mental control room. This sudden outburst of high-energy sensory bombardment creates pain, and the natural reaction of the brain is to grab hold of the levers of action (play around on the action-tangents) at random - in order to try to relieve it. In doing so, one of the first things that the brain discovers is that pulling the voice lever as hard as possible seems to generate the best possible form of relief - by the interaction with some strange external phenomenon later to be referred to as ‘mother’. In the beginning, the brain is able to stand this sensory monitored sound and light show only for short intervals without ‘blowing its tubes’. Therefore, during the first few months its main activities consist of shutting it out by sleeping and easing the pain of it by crying and eating. The situation is very similar to that which was facing the first men on the moon, who only dared to explore the external reality for a few hours each day, and spent the rest of the time hiding inside the familiar reality of the space-ship. And yet, the strangeness of the moon as it appears to an astronaut is small compared to the strangeness of the world as it must appear to a new-born baby. No wonder then if babies cry a lot.

## 10.3 The Internal Workshop

If you want to relate to yourself as a new-born baby, try to imagine yourself as being stuck inside the control room of your own ‘mental space-vehicle’ with all your sense-monitors going bezerk on you, and with no mental conception of any ‘outside’ at all. How would you go about sorting out the chaos of sensory impulses? Looking for the structure you need to find out how to manage your own controls; how to play your action piano in harmony with the sense-music that appear on the monitors. The name of the game is adaptation (= survival), because it is literally a matter of life and death.

To compose the necessary action-music for the adaptation game, each player makes use of his or her inherent abilities. In the human brain these gifts of nature include such abilities as *memory, pattern-recognition, abstraction, imagination* and *emotion*. These abilities are by no means independent or mutually exclusive of one another, and the division between them makes no claim as to being exact or complete in any way: In fact, they often overlap one another and work together in virtually inseparable ways. But they also possess a certain amount of individuality and inexpressibility in terms of combinations of the others, which merits a certain distinction between them.

We can think of these abilities of the brain as different kinds of materials and tools that are present inside the control-room. Taken together they constitute ‘the internal workshop’ where the brain is working to produce its action-music.

*Memory* can be regarded as a cabinet file where the brain can keep things stored away, available whenever it decides to bring them out again.

*Pattern-recognition* is basically the sensitivity of the brain to a variety of different forms and structures. It can be likened to the tool bord of a carpenter with different shapes - like e.g. ‘triangle’, ‘square’ and ‘circle’ - hanging from the nails instead of hammer, saw and screw-driver. The brain makes use of these shapes to identify patterns and make relevant groupings of the incoming sense-data. It seems that some elementary shapes are present on the tool-board from the very beginning, while the more sophisticated ones are added later, constructed by the brain with the help of memory, imagination and abstraction in accordance with the observations that it makes on the sense-monitors - just as a good carpenter constructs a new tool if the need for it arises often enough.

It is important to realize that many of the shapes and structures hanging from the mental tool-board are not visual or geometrical in themselves. We can ‘iconify’ them - and represent them to us as geometrical shapes - but since they relate to the non-visual part of the ‘sense-spectrum’, they must necessarily be of a non-visual nature.

Memory and pattern-recognition are abilities that we share with a lot of animals, but the ability to *abstract* is probably a unique feature of the human brain. *Abstraction* is the mental ability to recognize similarities in the behaviour of different groups of sense-data, and to shape these similarities into a concept of which the mind has no direct sensory experience, i.e. an ‘abstract concept’. Abstraction forms the basic foundation for human speech. Almost every type of verbal communication is carried out in terms of abstract concepts. We speak comfortably of e.g. a ‘chair’, and everybody understands what we mean by that word, but at the same time noone has ever had any kind of direct sensory experience of it. What we actually have experienced is thousands of different physical objects (groups of sense-data) of various shape, size, texture and colour - but with the one thing in common that we could sit on them. By the use of abstraction the brain recognizes the common behaviour-pattern of all these different phenomena and molds it into an abstract entity - a ‘chair’. If our brains were not able to perform this function, we would be forced to relate individually to everyone of the multitude of phenomena that constitute the ‘concrete raw-material’ for this un-experienced concept. This would undoubtedly place an enormous strain on our memories, since they would have to keep all this sensory information stored and available to us. We would literally have to remember each individual chair that we had ever encountered, in order to be able to make use of it when it appeared on our sense-monitors again. It is as good indication of the usefulness of the concept of ‘chair’ that it allows us not only to deal with an already experienced chair without remembering its individual characteristics, but it also enables us to walk up to a totally new sense-phenomenon (a chair that we never laid eyes on before) and make perfect use of it.

By performing the trick of abstraction the brain relieves its memory of a large amount of superflous information, thereby clearing a lot of memory-space which then can be put to better use. The reason why it is possible for us humans to control and manipulate animals may well be that they are too busy remembering things!

Within the setting of the internal workshop we can think of abstraction as a sort of ‘disintegration spray’, a very strong type of acid that the brain can apply to a group of phenomena in order to dissolve their individual sensory appearance and lay bare the core of their common structure. In short, abstraction is an excellent cleaning-fluid for the memory-cabinet.

The essence of *imagination* is opposite to that of abstraction - much as creation is opposite to destruction. In the mental workshop we can imagine imagination as a tybe of ‘creation gas’ capable of creating any new kind of concept or tool that the brain can conceive of. Whenever it wants a new concept ‘to help sort things out’ a bit better, the brain turns on the creation-gas which ‘makes the wish come true’. If the new tool is a good one - i.e. if it is helpful in composing better action-music - the brain will pin it on the tool-board. Otherwise the brain will get rid of it and try to create a better one. /////// Anaximenes ‘air’ (pneuma).

The most important function of imagination is to invent a ‘cause’ for each ‘effect’ that the brain observes on the sense-monitors. The notion of cause and effect is such a basic creation of the brain that it is difficult not to accept it as a fundamental truth. From the very beginning, the brain develops its action-music by repeating the individual chords over and over again. Eventually, the brain cannot avoid observing that a certain action-chord always seems to correspond to the same relevant sense-readings. Therefore it naturally assumes that the action-chord ‘creates’ the sense-readings, thus representing the action-chord as the cause and the corresponding sense-readings as the effect.

Approaching the problem of classification of sense-data, it is natural for the brain to start with the sense-readings that it can manipulate and reproduce internally. This leads to the establishing of the framework of ‘cause and effect’, and by the time the brain is ready to deal with sense-data which it cannot control in this way, the established pattern of regarding sense-readings as effects makes it natural for the brain to assume that they too must have a ‘cause’, something must make them appear on the sense-monitors.

To solve this problem, the brain makes use of imagination. It imagines the concept of an outside, an external reality beyond its own manipulative control, wherein it can imagine the causes of these sense-readings to be located.

The concept of an 'external world' ('disregarding oneself') is such an enormously useful tool for the classification of sense-data that very few people seem to seriously doubt its actual 'existence'. But it is important to understand that the concept itself - whether adequate or not - is a creation of the mind.

*Emotion* is the last of the mental workshop abilities on our list, and it is undoubtedly the most subtle and complex one of them all. It is extremely difficult - if not impossible - to 'metaphorize' because its basic structure does not lend itself to 'verbal dissection'. We can think of the faculty of emotion as being located in a separate part of the mental workshop - 'emotion-space' - as opposed to the other abilities which are located in 'thought-space' or 'rational space'.

We are instinctively aware that thinking and feeling are two structurally different ways of evaluating an experience - both of which are necessary aspects of understanding it. They are both governed by their own internal logic, and they each take place in their respective part of the workshop. The brain cannot cross the border-line between them without at the same time changing its mode of operation from 'thinking' to 'feeling' or vice versa.

Whenever it wants to 'think about its feeling' or communicate them verbally, the brain has to work in thought-space (using thought-logic) in order to construct a 'thought-space-model' of its corresponding 'emotion-space-configuration'. Since the basic texture of these two spaces are very different, this problem presents us with enormous difficulties, and we spend a lot of our time trying to figure out 'what we feel' and 'why we feel it'. The urge to analyze and communicate his feelings has been an integral part of the history of man, leading through the ages to the evolution of such emotional disciplines as poetry and art.

### 10.4 The Evolutionary Dilemma of the Human Brain

Concerning the problems of communication between thought-space and emotion-space, Arthur Koestler has ventured a very interesting and highly worrying analogy. He discusses the human dilemma, the violence-based interaction methods (= 'military solutions') that seriously threaten the long term survival of the human race, and traces its roots back to "the quasi-schizofrenic division between reason and emotion, between the rational intellect and the irrational and emotional belief-systems." [Janus, p.18]. Koestler refers the reader to MacLean[1], who has summarized the functioning of the human brain in this highly illustrative way:

> Man finds himself in the predicament of having been equiped by nature with basically three different brains, that have to work together and communicate in spite of substantial structural differences. The first (and oldest) one is a basic reptile brain. The second one has been inherited from the lower mamals, and the third one is part of a very late development, which has evolved into the specifically human part of the brain. Speaking metaphorically about these three types of brains within the brain, we could imagine that when a psychiatrist offers his patient to lie down on the coach, he offers him to stretch out on top of a horse and a crocodile.

1. MacLean, P. D., *Journal of Nervous and Mental Disease*, vol 135, #4, Oct. 1962.

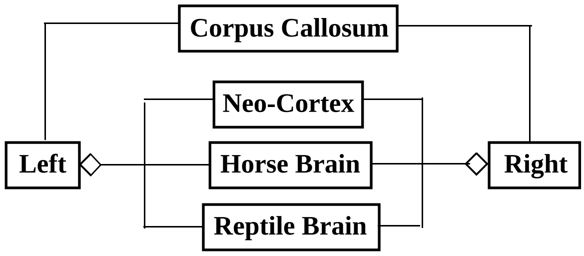


**Fig. 29. A schematic structure of the human brain.**

According to Koestler, if we substitute the single patient for humanity at large, and the psychiatrist's couch for the scene of history, we arrive at a grotesque but fairly accurate picture of the human condition. During another lecture in neuro-physiology MacLean made use of the following metaphor[1]:

> In the terminology of today, one could think of these tree brains as three biological computers, each one with its own special form of subjectivity as well as its own intelligence, its own feeling for time and space, its own memory, its own motorical system and so on.

Koestler goes on to describe the evolutionary history of the brain [(91), p.19)] in the following way:

> The reptile brain and the primitive mamal brain together form the so called 'limbic system', which we can refer to as the 'old brain' - as opposed to the *neocortex*, the specifically human 'thinking-cap'. [...]
>
> But while the antideluvian structures of the innermost parts of our brain - governing our instincts, passions and biological desires - have hardly been affected at all by the forces of evolution, at the same time - during the last half miljon years - the human neocortex has grown and developed at an explosive rate which lacks a known counterpart in evolutionary history. In fact, the growth of the human neocortex has been so rapid that some anatomists have compared it to a form of 'evolutionary tumour'.

Koestler points out that biggest evolutionary mistakes were made in the development of different kinds of brains. Hence, the invertebrates developed their brain around their intestinal tract, which ment that as the nerve-mass was developed and expanded, the intestinal tract would by necessity become more and more compressed. In fact, this is what happened to spiders and scorpions; their intestinal tract were strangled by their brain - an evolusionary design dilemma which they managed to survive by becoming blood-suckers. Koestler refers to Gaskell, who describes the situation thus[2]:

> During the process that lead to the appearance of the vertebrates, the evolutionary design and development of variations within the invertebrates (leddjuren) was leading up to a terrible dilemma, caused by the way the brain was pierced by the intestinal tract - either the ability to digest the food without enough intelligence to catch it, or enough intelligence for catching the food, but no ability to digest it.

1. MacLean, P. D., *A Triune Concept of the Brain and Behaviour,* Boag and Campbell (ed.) 1973.
2. Gaskell, *The Origin of Vertebrates,*

In the words of another eminent biologist, Wood Jones[1]:

> This markes the end of the evolutionary brain development among the invertebrates. They commited a fatal mistake when they started building the brain around the intestinal tract. Their efforts to develop a large brain failed [...] A new attempt had to be made.

Koestler then goes on to say:

> This attempt was made by the vertebrates. But a major branch of vertebrates, the Australian 'pungdjur' were also caught in an evolutionary form of 'cul-de-sac'. In contrast to us 'moderkaksdjur' the 'pungdjur' carry their undeveloped new-born in a 'pung'. Their brain lacks an important component, corpus callosum, - a distinct nerv-area which connects the left and right hemisphere of the brain of us 'moderkaksdjur'[2]. The neuroscientists have recently discovered a basic division of labour between the left and right half of the brain, which seem to functionally complement each other like Yin and Yang. Evidently, the two hemispheres must cooperate in order for the mamal (or the human) to make full use of their capacity. Hence, the lack of a corpus callosum indicates an 'insufficient coordination' between the two parts of the brain - an expression which sounds ominously familiar. This could well be the main reason that the evolution of the 'pungdjur' - in spite of bringing forth many species that carry striking resemblances to their cousins among the 'true mammals' - came to a halt at the present level of the koala.

According to Koestler, the never ending catastrophies in the history of man are mainly due to our grossly exaggerated ability to identify ourselves with some form of tribe, nation, church or cause - and to accept its credo uncritically and enthusiastically, although these 'wholy truths' are in blatant contradiction to both our reason and our self-interests and even substantially deminish the chanses of our personal survival. In [(91), s. 23] he concludes:

> Hence we are brought to the immodern conclusion that the problem with our species is not an excessive amount of aggression, but an exaggerated ability of fanatical devotion. Already a quick glance at history should convince us that the individual crimes that are committed 'for selfish reasons' play a very subordinate role in the grand human tragedy - compared to the masses that are butchered in unselfish loyalty towards a tribe, nation, church or political ideology *ad majorem gloriam dei*. Except for a small minority of greedy or sadistically inclinded individuals, no wars are fought for personal gain, but for the cause of loyalty and devotion to the king, the country or the cause. Murder for selfish reasons is the statistical exception in all cultures, including our own. Murder for unselfish reasons, with a grave risk for one's own life, is the totally dominating form of historial expression. [...].
>
> Man's most deadly weapon is his language. He is as susceptible to the verbal hypnotism of slogans as he is to infectious diseases. And when such a verbal epidemic strikes, the group-soul takes over. It obeys its own rules that are different from the behaviour protocols of each individual member. When a person identifies with a group, it decreases his ability to reason and increases his passions through a kind of emotional resonance or positive feedback. The individual is no killer, but the group is, and by identifying with the group the individual is transformed into a killer. This is the 'diabolic dialectic' which is reflected in the human history of wars, persecution and genocide. And the foremost catalyst in this metamorphosis is the hypnotic power of words. Adolf Hitler's words were the most important destructive weapon in his time. And long before the printing press was invented, the words that were spoken by Allah's chosen profet started an emotional chain-reaction which shook the world from central Asia to the atlantic coast. Without words there would be no poetry - and no wars. Our language constitutes the main strategic factor behind our feelings of superiority towards our animal brothers, and - in view of its explosive emotional power - it poses a constant threat to our survival.

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1. Wood Jones, F. & Porteus, S. D., *The Matrix of the Mind*, Habit and Heritage, London 1943 (1920).
2. Mera exakt, de högre (icke-olfaktoriska) funktionella områdena.

# 11 The Concept of a Knowledge Manifold

The urge to classify and structure knowledge is an ‘intellectual force’ that has been felt by man - and responded to in a multitude of ways - for at least 2500 years according to written record. It flourished in the magical 6:th century BC, when men like Thales from Miletos and Pythagoras from Samos were shaping the foundations of the mathematically based, scientific way of thinking about the world that dominates us today. The revolutionary new idea that swept the Greeks at this time was the following: *The world can be intellectually understood!* It is not the whimscal playground of unpredictable gods, but a *reasonable* structure - more like a kind of mechanical machine - operating according to well defined and explicitely formulatable “natural” laws! This marks the beginning of the *age of reason*, the *“rational project”*, which lasted for about 300 years - after which time it was put on ice and reawakened almost 2000 years later by the intellectual curiosity of the renaissance.

## 11.1 Epistemological Framework

On of the most fundamental questions that anyone can ask is “How do I know that what I think I know is true? The study of this question is called *epistemology.* It is a complex subject, one to which many philosophers and theologians have devoted their entire careers. The discussion here will by necessity be brief and incomplete, but should suffice to demonstrate the critical importance of the subject to software engineering in general, as well as to the conceptual structure of the GOK in particular.

The GOK is an example of an epistemological framework that I have termed a *Knowledge Manifold*. In any type of Knowledge Manifold, the representable world is regarded as consisting of a number of *objects*(= entities) of type *phenomenon* (=*percept*), *concept* or *tool*.

**Definition 20:** By a *phenomenon*, I mean a group of sense-perceptions that ‘stand out’, i.e. that have a well defined boundary (= crisply defined interface) with respect to the rest of the world as perceived by some observer.

A phenomenon represents groupings of the observer´s perceptions that are internally *coherent*, i.e. that occur together in some strongly coupled way. *Light*, *sound*, *heat* and *electricity* constitute examples of *physical phenomena*.

The structure of a Knowledge Manifold is based on the following fundamental

**Definition 21:** An *idea* (= *concept*) is a representation of an *experience*.

This represents a totally anti-Platonian definition of the concept of idea. Plato´s ideas are eternal and unchangable ‘basic structures’, while the ideas of a KM are subjectively based, and originate in the individual’s desire to represent his own experiences.

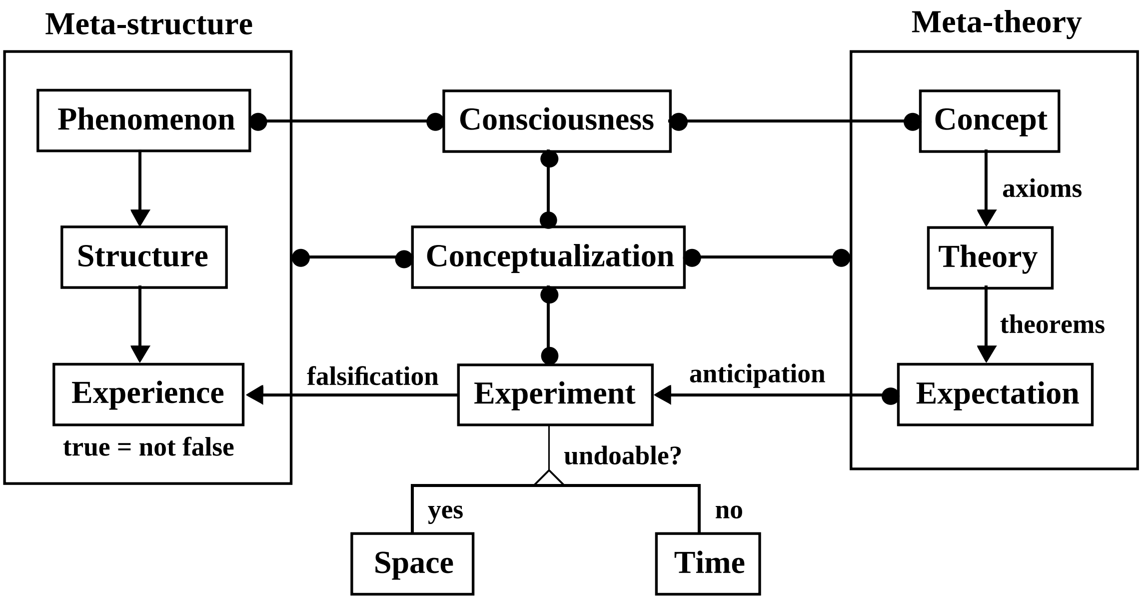


**Fig. 30. The Meta-structure of a Knowledge Manifold**

*Concepts* are mental constructions whose purpose is to provide structure to the manifold of phenomena that we experience, thereby making these phenomena as well as their mutual relations more graspable, manageable and describable to the human intellect. The concepts render the phenomena *speakable* to us in the sense of Wittgenstein[1]. Some concepts are direct linguistic images of phenomena, as e.g. the concept of light - while other concepts are more abstract and lack a direct coupling to a certain phenomena, as e.g. the concept of energy.

*Concepts* (= *ideas*) come in two basically different kinds, *inward ideas* are expressed mentally and *outward ideas* are expressed medially. The *mental concepts* are private and can be used only for personal speculation while the *medial concepts* are are communicable to others. The medial ideas in turn occur in different forms, mainly *text*-, *image*-, *sound*- or *motion*-based. Digital mixtures of such concepts are presented today under the familiar label of *multi-media*..

In a Knowledge Manifold, relations between phenomena and concepts are represented by various kinds of *tools*. Using an *active tool*, one can perform experiments of different kinds, while a *passive tool* conveys information through different types of sensing media, most often visual images, sound or text.

Your consciousness *experiences* phenomena, *creates* concepts and *experiments* with tools. The phenomena are related directly to your experiences, the concepts are related to your efforts to describe and interpret these experiences, and the tools are related to the experiments that you perform with this purpose.

1. Tractatus Logico-Filosoficus [(174), p. ???]

Aquiring knowledge is to build good (= useful) descriptions of various kinds of phemomena and their multitude of possible interactions. Such descriptions are often called *explanations* or *theories*. In order to explain the intriguing interplay of phenomena, we form concepts in our minds, concepts that we then combine and relate to each other deductively (by "logical reasoning" or "theorizing"). In physics, to take an example, the phenomenon of light can be described theoretically by using concepts like time, energy, particle, wave or photon. The power of a group of concepts lies in the usefulness of the theories that can be generated on top of them. Each theory is based on a number of concepts that it cannot explain within itself. Such concepts will be referred to as *basic*. Basic concepts are used within a theory as the building blocks for an explanation of its other concepts. Hence, within the theory of physics, we cannot explain concepts like *force*, *mass*, or *energy* - anymore than we can explain the concepts of *point*, *line* or *plane* within the theory of geometry. Descriptions such as "two points determine one line" and "two lines determine one point" do not tell us anything about the nature of a point or a line in itself. Here we encounter a clear-cut example of what could be termed the "Wittgenstein unspeakability principle". we formulate this as a

**Fact 15:** The *basic concepts* [1]of a theory mark its boundary towards the unspeakable. Only the *relations* between its basic concepts are *expressible* within its formal language.

Poincaré has formulated the same idea in the following words:

> The aim of science is not things themselves - as the dogmatists in their simplicity imagine - but the *relations* between things. Outside those relations there is no reality knowable[2].

Theoretical descriptions (expressions) are usually divided into two separate classes: *axioms* and *theorems.* Both axioms and theorems are speakable (expressable) within the theory, but only theorems are provable (deducable) within it. The axioms of a theory represent its assumptions and the theorems of the theory represent its most important deductions[3]. To be provable within a theory means to be deducible from its axioms and its already established theorems, applying nothing but the theory´s own logic, i.e. its own rules of deduction[4] (= deductive inference).

The mind works part deductively, by theorizing - part inductively, by collecting experiences. A theory creates expectations that often lead to experiments, the outcome of which in turn can verify or contradict the corresponding theoretical predictions, thereby influencing the conceptual evolution of the theory itself. In this way the concepts of *theory* and *experiment* (≈ *deduction* and *induction*) form together a dynamic pair of counterparts, a *conceptual duality* similar to the classical *yin-yang* pair of ancient Chinese philosophy.

1. Within the theory, we can talk *with* them (= using them), but not *about* them.
2. [(128)], p. xxiv.
3. A deduction of a minor nature is often referred to as a *lemma*, if it forms part of the resoning steps leading up to a theorem, or a *corollary* if it appears as a natural consequence of some theorem(s).
4. An example of a deduction-rule within the theory of *aristotelian logic* is given by the familiar *modus ponens*.

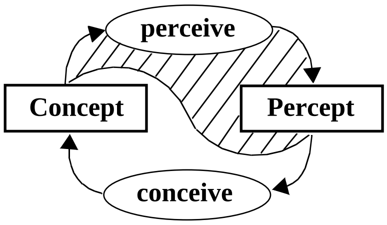


**Fig. 31. The Yin-Yang duality of the Perception-Conception cycle.**

Referring back to the ‘philosophical simulator’ pattern of Figure (30), experimental testing of intellectually conceived and formulated theories represents the scientific way of relating phenomena and concepts. According to Popper, we build up theoretical (deductive) expectations and collect experimental (inductive) experiences by trying to falsify our theories as best we can. In fact, the experimental results are based on measurements where the concept of equality is defined in terms of non-detectable difference[1]. “An experiment is a falsification-machine for theoretical expectations” - Popper could have stated the matter. Whatever does not surrender to falsification eventually aquires the status of scientifically established fact, or *“law of nature”*. In fact these theoretical truths, these *natural laws*, are not like universally established laws in the sense of totally immutable truths. In fact, they are more like *intellectual tools*, a sort of mental wrenches and screwdrivers with limited power and range of application. The laws of nature provide us with thought-recipies for cooking up theoretical expectations regarding a lot of different physical configurations.

## 11.2 Concept Geometry - Linguistically Based Concept Formation

In his famous Erlanger program of 1872, the great geometer Felix Klein proposed to define a geometry as a collection of statements concerning ‘objects’ that remain invariant under a group of transformations. This marks the beginning of the modern viewpoint - where each geometry is regarded as a sort of language, with its own collections of *transformations* (= *verbs*) and *invariants* (= *nouns*).

Our linguistic structures (languages) are intimately related to our methods of concept formation. The *verbs* describe the *operations* (changes) that we can observe - or imagine - while the *nouns* describe the *invariants*, i.e. the substances that *survive the operations* (*transformations*) of the verbs. The *adjectives* describe *values of aspects on the nouns*, like in the phrase “the red car stopped” where the adjective ‘red’ represents a value of the aspect ‘colour’, which can be associated with the noun ‘car’

This situation carries strong similarities to geometry. A geometric theory behaves like a language: It has its own verbs, that express the types of motions that the geometry admits, and its nouns that express its invariants, i.e. the concepts that survive the motion-transformations of the verbs. To give an example, the concept of *square* is a noun of *euclidean geometry*, because the transformations of this geometry consist of ordinary (rigid) motions, reflections and uniform

1. Compare the discussion of Chapter (12).

scalings, and each of these types of operations transforms a square into a square - "leaves the squareness invariant", as a mathematician would put it. 'Square' is thus a euclidean geometric concept, because it survives the action of the euclidean verbs. However, in the world of geometric shadows, *projective geometry,* the concept of ''square' is not a valid noun. Projective geometry considers *all shadows equivalent*, and its verbs (motions) are represented by combinations of shadow-projections. Projective geometry therefore has no room for squares, since the shadow of a square - cast from a point outside of its plane onto another plane - is not in general a square, unless the planes happen to be parallel in space. Hence a square does not survive the (=*all*) actions of the projective verbs, and 'square' is therefore not a projective noun (= invariant)[1]. This is illustrated in Figure (38), which describes the multi-layered structure of geometry, regarded as a so called *knowledge-component* [see Chapter (11.6)].

## 11.3 Concept Classification

The basic epistemological structure of a knowledge manifold is shown in Figure (32). The concepts are divided into *containers* and *elements*. A *container-concept* consists of a number of *concepts* and *tools*. It has an associated *map*. An *element-concept* is classified as an *operation*, a *substance* or an *aspect*, in accordance with the concept geometry - as described above.

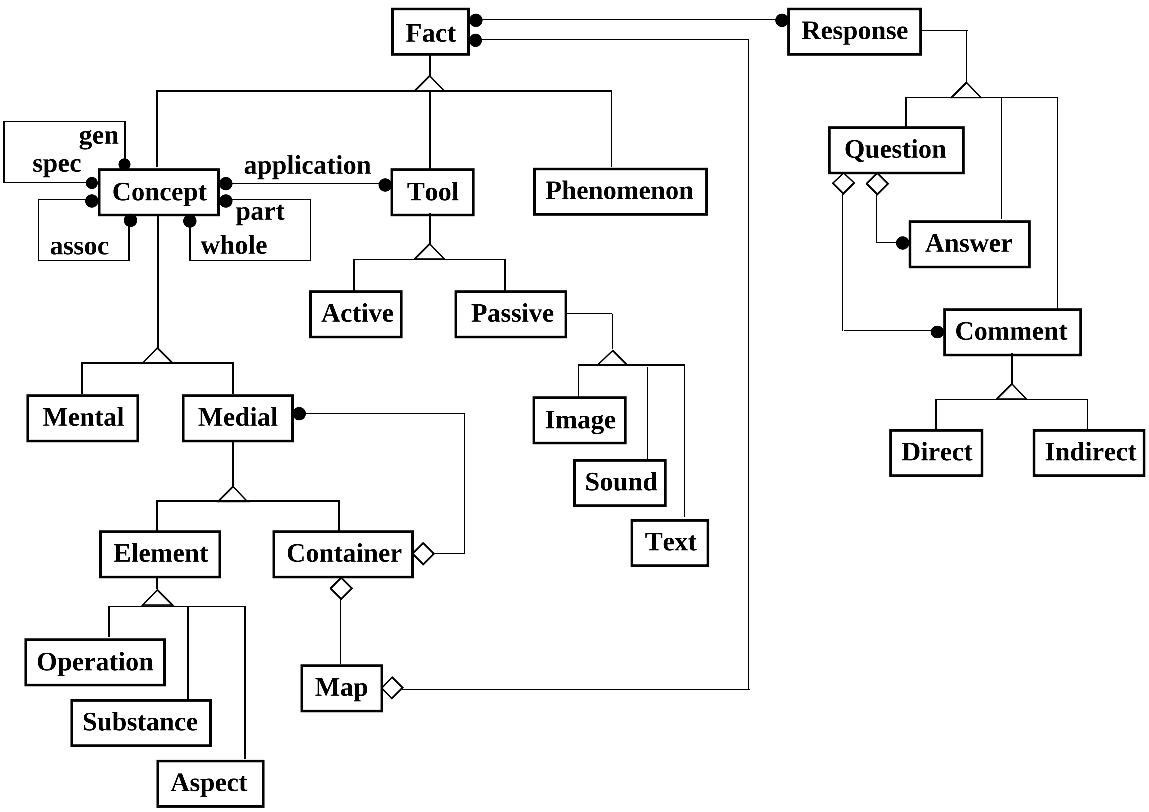


**Fig. 32. Conceptual classification of a Knowledge Manifold**

1. Since a projective transformation can take any four (generically placed) points into any other four-tuple of such points [see e.g. (22)], a square can have no special structure in projective geometry - apart from its 'planarity' and its '4-edge-ness'. Hence, in projective geometry, the word 'square' means just a general *quadrangle* (or *quadrilateral*).

An *operation-concept* corresponds to a *verb*, and describes a process of change (transformation). A *substance-conscept* corresponds to a *noun*, and describes something that remains unchanged (invariant) during the transformational changes of the verbs. An *aspect-concept* evaluates[1] to an *adjective*, and describes some type of invariant property of a substance-concept.

As an example from ‘everyday life’ let us consider the fact that ‘middle-aged men jog’. Here middle-aged men are subjected to the action of jogging, but they remain none the less both men and middle-aged (at least in the short run). “Middle-aged men are invariant under jogging”, as a mathematician would put it.

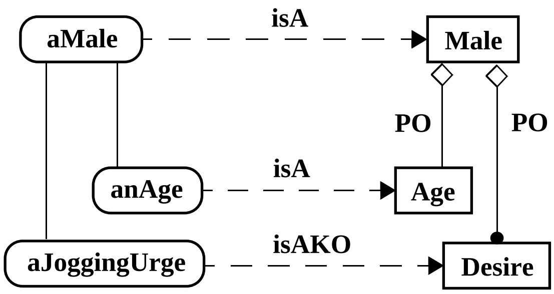


**Fig. 33. Middle-aged male jogging modeled as a kind of desire.**

It is important to emphasize that our linguistic flexibility makes it impossible to fixate a concept classification scheme along the lines of any kind of word-type-classification. Any such scheme is bound to contain an element of *‘arithmomorphic distortion’*, to speak in the terms of Georgescou-Roegen[2]. In the previous example, when we study the fact that “middle-aged men jog”, we would probably be more inclined to say that we study “middle-aged male jogging”, as in Figure (33) or perhaps “jogging, middle-aged males”, as in Figure (34). Note that both of these linguistic transformations of the initial statement are nouns. In theoretical science *‘nounification’* is a very common strategy of conceptualization[3]. Whenever we classify and structure concepts we often tend to reformulate linguistic configurations by inventing new nouns or adjectives that we glue together into some more or less tangible thing.

1. As, an example, the value of ‘colour’ might be ‘red’.
2. He refers to the process of introducing sharp *arithmomorphic* boundaries (of type either / or) when describing soft *dialectical* concepts (of type ‘day’ and ‘night’) which have no sharply defined boundaries. [(57)].
3. This is natural, since we tend to be much more familiar (and comfortable) with classifying *things*, than with classifying *processes.*

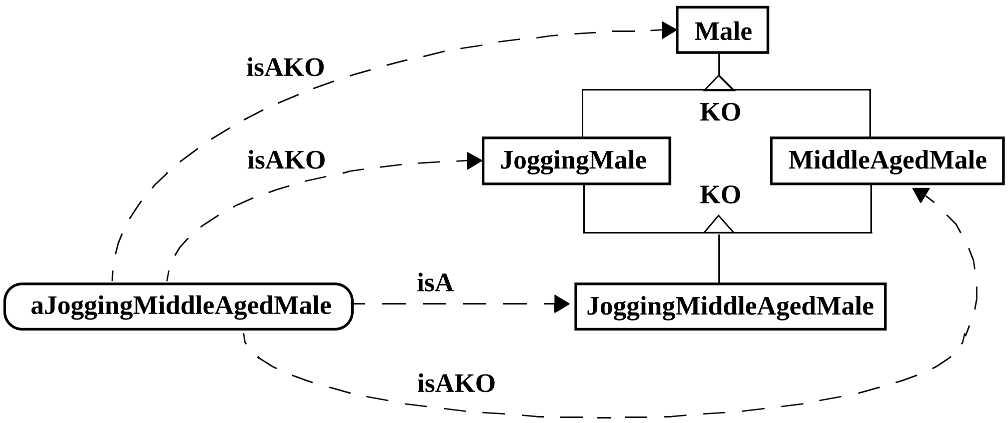


**Fig. 34. aJoggingMiddleAgedMale modeled as a kind of male.**

The concept of ‘jogging’ is a good example. When we talk about ‘middle-aged male jogging’, ‘jogging’ is a noun - presumably invented from ‘jog’. The parts ‘middle-aged’ and ‘male’ are two aspect-values[1] of the substance-concept ‘jogging’. However, when we talk about “jogging, middle-aged males”, ‘jogging’ is an adjective invented from the same source. It now appears as an aspect-value of the substance-concept ‘male’.

This exposes the  difficulty of any tree-like (single inheritance based) classification scheme: ‘middle-aged male jogging’ is a sort of jogging, whereas ‘jogging, middle-aged male’ is a sort of male. When we classify the fact that ‘middle-aged men jog’ we must take care to differentiate clearly beteen these two cases. We may want to concider this fact either as a kind of jogging or as a kind of male, or maybe as a kind of both - just as a ‘houseboat’ can be considered as a kind of ‘house’ and a kind of ‘boat’..

The jogging example above is ment as a motivation for the fact that a concept in a KM can be part of several type hierarchies (multiple type-inheritance). We will return to these questions below.

## 11.4 Classification of Responses

In the terminology of the GOK, Students are invited into the teacher´s garden and given the opportunity to respond to various types of configurations questions, some of them “defaulted” by the teacher-gardener, others constructed by the students themselves. A *response* can be of type *question*, *answer* or *comment*, and a *comment* can be either *direct*, i.e. expressed in *multimedial form*, or *indirect*, i.e. expressed in the form of a *link* (*adress*)

Questions can, of course, be infinitely varied, and the student must therefore be encouraged to formulate his own variations.

1. ‘Middle-aged’ is the value of the aspect ‘age’, and ‘male’ is the value of the aspect ‘sex’.

Following our language-based conceptual classification scheme, it is natural to use the *interrogative pronouns* as a base for the *various subtypes of question*. This means that each question will fall into one of the following basic categories shown in Figure (35).

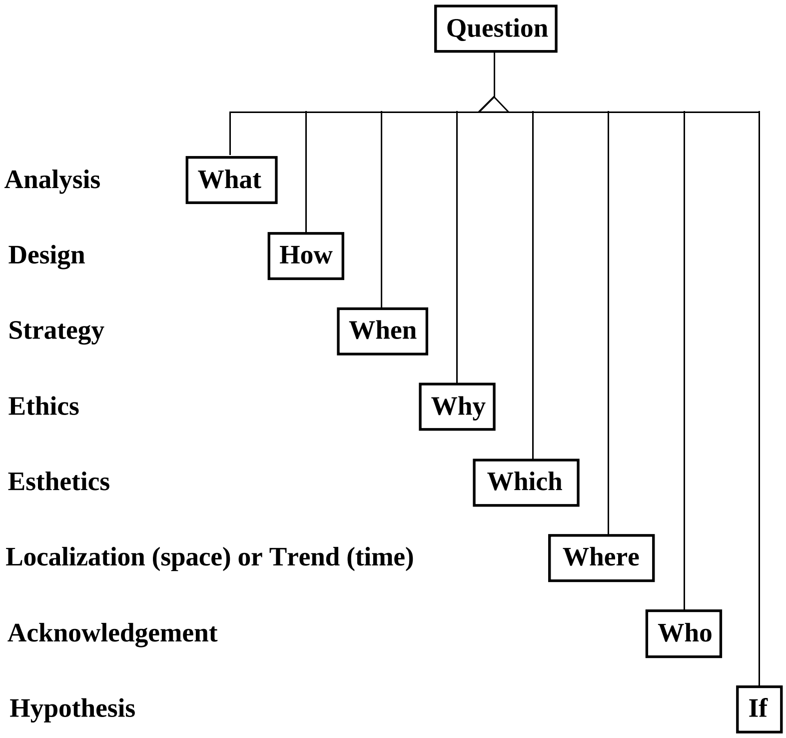


**Fig. 35. The basic questions of the interrogative pronouns**

The question of WHAT (*analysis*), HOW (*design*), WHEN (*strategy*), WHY (*ethics*), WHICH (*esthetics*), WHERE (*localization* or *trend*), WHO (*acknowledgement*), IF (*hypothesis*).

These subtypes are used as building blocks to construct the *well-formed questions* within the GOK. Two familiar examples are given by the WHY-WHEN[1] or WHAT-IF[2] type of questions. The former type of question is considered here as a subtype of WHY, and the latter type as a subtype of WHAT.

The students should be free to comment on the teacher's 'root-responses' or 'default-responses' to these questions. When the student is engaged in this activity, any pre-recorded root-responses of the teacher-gardner should be available, making it possible for the student to follow these *explanatory comments* - in either *direct* (=*multi-medial* ) form or *indirect* (=*link* ) form - further into the literature and onto the net.

Recall from above that a *question* is defined as a form of response to a combination of concepts. When a student selects some combination of concepts and formulates a question, the latter is

1. WHY are you angry WHEN I eat my porrige.
2. WHAT-IF I didn´t eat it, would you still be angry?

interpreted as related to the *selected concepts* (the *current context*). If no one has asked the question in this context before, the student is given the opportunity to store it together with his or her own response. If the question has already been responded to - by the teacher/gardener or by some group of co-workers in the GOK - these responses are made available to the student, who can then add and store his own reponses 'on top of' the present configuration.

## 11.5 The Relations between Concepts

Knowledge is always organized and structured, concepts always formed and related to others by dividing and classifying the world of appearances according to some form of *a priori* chosen epistemological scheme. The epistemological model chosen for the GOK supports three different types of *relations* between concepts: *abstraction/concretization* [1], *ingredience/context* [2], and *association* [3].

Consider two concepts denoted respectively by *A* and *B*. If *A* is considered to be an *abstraction* (=*generalization*) of *B*, then we must at the same time regard *B* as a *concretization* (=*specialization*) of *A*. In the jogging-example above, the concept of 'male' can be considered (=modelled) as a natural abstraction of the concept of 'middle-aged male', which in turn can be viewed as a 'pretty natural' abstraction of the concept of 'jogging middle-aged male' - if the jogging-obsession is strong enough. From the reverse perspective, the concept of 'male' is then subjected to *specialization* (=*concretization*) - first into the concept of 'middle-aged male', and then further, into the concept of 'jogging middle-aged male'.

Consider two other concepts, called *C* and *D*. If *C* is an *ingredient* (=*member*) of *D*, then *D* is a *context* (=*aggregate*) for *C*. Returning to the jogging example, we can just as well choose to regard the concept of 'middle-aged' and the concept of 'jogging', as two *ingredients* of the concept of 'jogging middle-aged male'. The latter concept then becomes a *context* (=*aggregate*) for the first two. Whether to go along with this new way of describing the relation between the concepts or to stick to the old is an epistemological decision, whose outcome depends on the circumstances.

The important difference between these two ways of concepual classification is that first way (*abstraction/specialization*) creates a *type-hiearchy*, often referred to as a '*Kind-Of*' - hierarchy, while the second way (*ingredient/container*) creates an *aggregation-hiearchy* (*container-hierarchy*) often referred to as a 'Part-Of' - hiearchy.

No matter which way we choose to look at it, we introduce a 'thought-pattern', a kind of 'mental grid' which was not there before. This is the essence of the 'speakability problem' Whenever we supply some kind of mental structure in order to tighten our reasoning about a problem the inevitable *arithmomorphic distortion* sets in.

As mentioned above, the GOK supports a third type of relation between concepts, namely associations (=links). The associations of a concept represents its cross-connections to other conceptual hierarchies.

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1. also called the *generalization/specialization* relationship.
2. also called the *member/aggregate* relationship.
3. also called the *link* relationship.

Concepts are related to tools by *application*. The applications of a concept are the programs (= tools) that are accessible (= can be applied) at the level of the concept itself. We can summarize the relations between various forms of concepts and tools in the following

**Fact 16:** A *concept* can be said to *aggregate* over its *ingredients*, *refer* to its *associations*, *concretize* (or specialize) its *abstractions* and *abstract* (or generalize)its *concretions.*

## 11.6 Building Knowledge-Components and Learning-Strategies

### 11.6.1 Conceptutal Overview and Design Goals

Returning to the question of designing educational environments, I will illustrate how the epistemological ideas developed above - especially concept geometry - can be used as a tool to describe the structure of such environments. Within the software engineering community the so called model-view is a well-known design pattern that involves separating the logical and the presentational aspects of a system. The advantage of such a separation lies in the increased modularity (= independence) between these two aspects, which means that changes in each aspect has smaller effects on the other than if the two aspects would have been more tightly coupled. Here is an interesting way to apply the model-view pattern to the design of interactive educational environments:

**Opinion 13:** It is desirable to separate between "what to teach" (= the factual model) and "when to learn" (= the presentational view). The increased modularity that follows from such a separation makes it possible to design interactive educational environments with a higher potential for individualized learning.

To make the discussion more clear, I will introduce

**Definition 22:** The attempts to answer the questions of *"what to teach"* and *"when to learn"* will be referred to as designing a *knowledge-component*, respectively a *learning- strategy.*

The following discussion is closely related to the educational paradigm shift that is described in Chapter (6.2), and which includes changing from the 'TeacherTenure-LearnerDuty' pattern of Figure (8) and adapting to the 'TeacherResource-LearnerRight' pattern of Figure (9).

**Opinion 14:** The traditional, collectively oriented, class-based curriculae should be replaced by individually oriented curriculae, i.e. learning-strategies based on each individual and his or her unique interests and capabilities. It is only then that we can achieve the class-less educational system, or, as I prefer to call it *a first class educational system for all*.

When we stop to think about it, the school is the only place where we consider a move from first to second class to be a form of improvement. Just imagine being treated the same way by e.g. an airline or a train company!

**Opinion 15:** As a student, I should not be given the feeling that 'I have to learn what the teacher already knows' (teacher-filter). Instead, I should be encouraged to feel that the teacher is an important resource in helping me to find out 'what I want to know'.

This is the overall purpose of the three different 'teaching-dimensions' of Figure (10) - the teacher as a *preacher*, *gardener* or *plumber*. The *teacher-gardener* helps to raise questions (by assisting each pupil in designing his or her own personal learning strategy), the *teacher-plumber* sees to it that as few questions as possible are lost, and the *teacher-preacher* sees to it that at least some of the questions are answered.

Some - or even all - of these different roles can coexist within the same physical person, but with today's heavy demand for education, there just are not enough such 'super-teachers' to cover the need. One of the main advantages of a virtual learning environment is the fact that these three teaching-roles can be distributed across physical space, thereby taking advantage of the facilities of cyber space to act as an electronic switch-board, connecting curiosity and interest on the side of the learners with knowledge and communicative skills on the side of the teachers. In this way we can hope to overcome the devastating effects of the knowledge-filter discussed in Opinion (15).

#### 11.6.2 Designing a Learning Strategy

Designing a learning-strategy involves coming up with an answer to three important 'whats', namely *'What am I interested in?'*, *'What is there to know about it?'*, and *'What do I want to know about it?'*. Naturally, the answers to these questions are interconnected and mutually dependent in various highly complex ways. Nurturing these questions in an ongoing dialogue with each learner is an important task of the teacher-gardener.

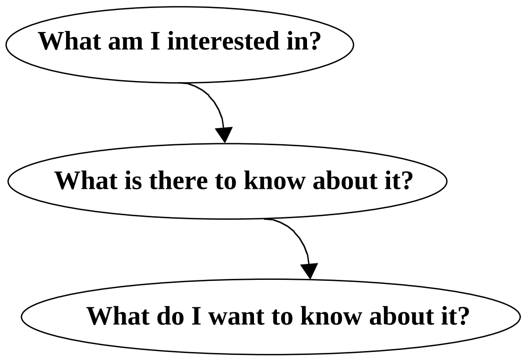


**Fig. 36. Three fundamental questions in desiging a Learning-Strategy.**

#### 11.6.3 Structuring a Knowledge Component

The conceptual framework presented above can be used in order to describe the structure of knowledge components as well as learning-strategies. A well designed knowledge component can be likened to a skiing area with many ways to get down from the top. Just as a good ski-pist is marked with a special colour - signalling its corresponding level of difficulty - a well designed knowledge component should have different entries - with some form of marking - corresponding to documented differences in prerequisite knowledge.

**Definition 23:** Individually oriented teaching/learning strategies are characterized by the following idea: Let the teacher decide *what* should be taught, and let the student decide *when* it should be learned.[1]

Separating between knowledge-components and learning-strategies is complicated by the fact that any given answer implies a pre-judgement of the context within which the question was formulated. This creates a 'strong dependency' between *what* is being presented and *when* (= under what kind of circumstances) it is being presented, which represents a hardwired relationship between *knowledge-content* and *learning-strategy.* A well-designed knowledge-componente avoids assumptions as to the context within which it will be used.

I will illustrate with an example from the third GOK prototype described in the group-report on the GOK-project. The geometrical knowledge-component called 'symmetry' contains a *milk-drop-movie*, as an example of the concept of 'symmetry breaking' and a movie on *corn circles*, illustrating the concept of rotational symmetry and *Curie's Principle* [see Chapter (7.3)]. In the bottom part of Figure (37) this structure is described as a knowledge-component with two different entries.

In the upper part of Figure (37) is shown an embryo of a learning-strategy - or rather one of the diagnostic questions that are involved in designing such a strategy - more specifically in deciding on what level to interact with the symmetry-component. Depending on the answer to the diagnostic question "Are you familiar with rotational symmetry?" the learner is presented with a suitable entry into the component - indicating that it you don't know about rotational symmetr, then you should study this concept before opening up the milk-drop movie, and a good place to start is the movie on corn-circles.

Before interacting with more complex types of knowledge-components, this kind of diagnostic-could be performed by each learner, and by logging the individual responses, the program can adapt to each individual and present a suggention of paths that are suitable for his or her personal traversal of the corresponding knowledge-component.

Such diagrams tend to become rather compex and hard to survey in two dimensions. It is an interesting area of future research to investigate various forms of 3-dimensional embeddings of such diagrams. The have the potential to present the structure of a subject in such a way as to increase both the 'conceptual clarity' as well as the 'navigational possibilities' within the corresponding 'information space'.

Concept geometry can be used to give a multi-layered concptual structure of a knowledge-component. In Figure (38), the geometrical example of Chapter (11.2) is expressed and expanded in this notation.

1. Since the teacher is responsible for the teaching, and the student is responsible for the learning.

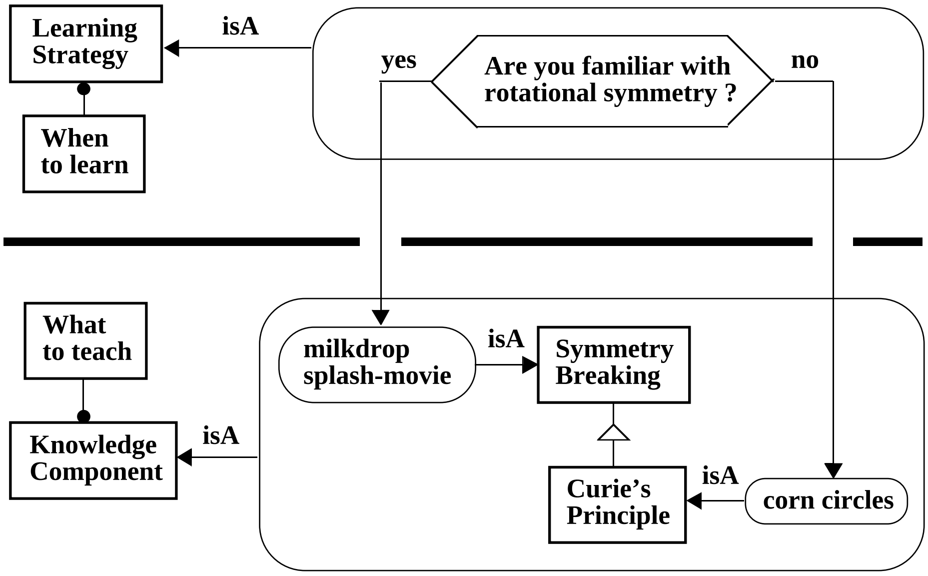


**Fig. 37. Interaction between a Knowledge-Component and a Learning-Strategy**

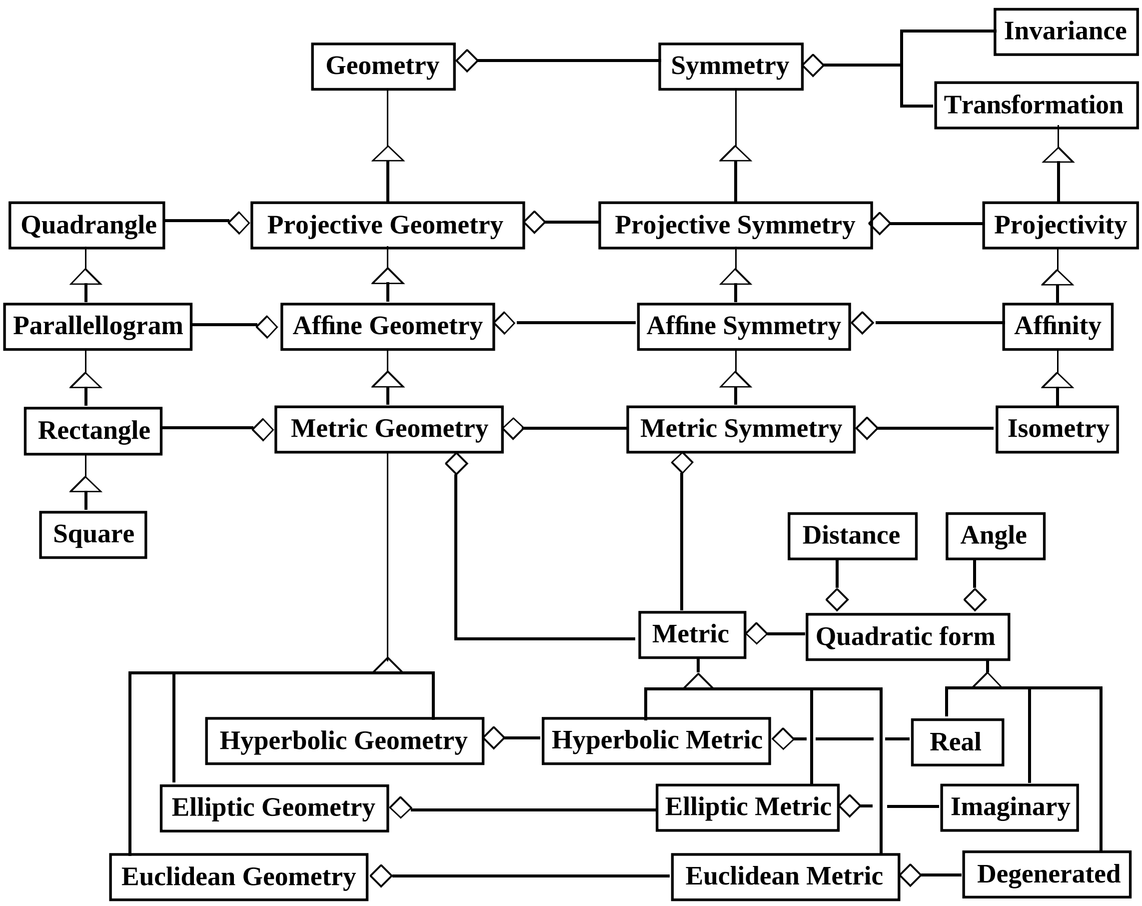


**Fig. 38. Part of the description of geometry as a knowledge-component.**

### 11.7 Structural Evolution of Knowledge

Each conceptual classification scheme is by its very nature dynamic and is therefore subject to evolutionary changes in the form of both refinements and paradigm-shifts. The concept of knowledge-evolution represents partly the formation of new concepts, and partly the 'transformation by refinement' of the old ones. The history of science provides many examples of this evolutionary process. Within the field of physics, the concept of 'atom' a hundred years ago used to refer to some 'entity' that was 'indivisible' (= had no parts). Hence, in these days, 'atom' could be described as a *substance-element-concept* [compare Figure (32)]. As atomic physics progressed, the concept of 'atom' was transformed into a *container-concept*, which inititally came to aggregate over (= form a context for) the substance-element-concepts of 'electron' and 'nucleus'. By the evolution of nuclear physics, the concept of 'nucleus' was in turn transformed into a container-concept, aggregating over such substance-element-concepts as 'proton' and 'neutron' - and eventually a whole bunch of other so called *'elementary particles'*.

### 11.8 The Research Front - The Surface and Volume of Knowledge

In a system where the only thing that counts (in terms of merits) is the contact with the surface, people will tend to neglect to take care of the volume. Let me state this as a formal

**Opinion 16:** In academia there is a problem to maintain the knowledge-volume, which grows as $r^3$, while earning your credits at the knowledge-surface, which grows as $r^2$. This tends to promote superficiality of research.

Bodil Jönsson has phrased it the following way:

> We devote an enormous amount of skills and energy to producing neat little research-parcels - containing useful and important factual knowledge - and wrap them up in neat little reports that are stored on the shelf and referred to by later publications - but never actually used by anybody. [1]

1. Prydliga forskningspaket som vi lägger en enorm kraft och energi på att ta fram, och sedan slår in i små rapportpaket och lägger på hyllan för att aldrig användas.

# 12 Establishing Equality by Disregarding Differences

## 12.1 Disregarding the Subjective Observer (= Oneself)

In his brilliant lectures, Schrödinger asks the fundamentally important question: What are the peculiar, special traits of our scientific world-picture? About one of these fundamental features, (he goes on to state) there can be no doubt, It is the hypothesis that *'the Display of Nature can be understood'*. [1]. It is the non-spiritistic, the non-supersticious, the non-magical outlook. A few pages later, Schrödinger continues[2]:

> There is, however, so I believe, a second feature, much less clearly and openly displayed, but of equally fundamental importance. It is this, that science in its attempt to describe and understand Nature simplifies this very difficult problem. The scientist subconsciously, almost inadvertently, simplifies his problem of understanding Nature by disregarding or cutting out of the picture to be constructed, himself, his own personality, the subject of cognizance.
>
> Inadvertently the thinker steps back into the role of an external observer. This facilitates the task very much. But it leaves gaps, enormous lacunae, leads to paraxoxies and antinomies whenever, unaware of this initial renunciation, one tries to find oneself in the picture or to put oneself, one's own thinking and sensing mind, back into the picture.
>
> This momentous step, cutting out oneself, stepping back into the position of an observer who has nothing to do with the whole performance - has received other names, making it appear quite harmless, natural, inevitable. It might be called just *objectivation*, looking upon the world as an object. The moment you do that, you have virtually ruled yourself out. A frequently used expression is *'the hypothesis of a real world around us'* (*Hypothese der realen Aussenwelt*). Why, only a fool would forgo it! Quite right, only a fool. None the less it is a definite trait, a definite feature of our way of understanding Nature - and it has consequences.
>
> The clearest vestiges of this idea that I could find in ancient Greek writing are those fragments of Heraclitus that we have been discussing and analyzing just before. For it is this 'world in common' this ξυνον or κοινον of Heraclitus, that we are constructing; we are hypostatizing the world as an object, making the assumption of a real world around us - as the most popular phrase runs - made up of *overlapping parts of our several consciousnesses*. And in doing so, everyone willy-nilly takes himself - the subject of cognizance, the thing that says 'cogito ergo sum' - out of the world, removes himself from it into the position of an external observer, who does not himself belong to the party. The 'sum' becomes 'est'.
>
> Is that really so, must it be so, and why is it so? For we are not aware of it. I'll say presently why we are not aware of it. First let me say why it is so.
>
> Well, the 'real world around us' and 'we ourselves', i.e. our minds, are made up of the same building material, the two consist of the same bricks, as it were, only arranged in a different order - sense perceptions, memory images, imagination, thought. It needs, of course, some reflexion, but one easily falls in with the fact that matter is composed of these elements and nothing else. Moreover, imagination and thought take an increasingly important part (as against crude sense-perception), as science, knowledge of nature, progresses.
>
> What happens is this. We can think of these - let me call them *elements* - either as constituting mind, everyone's own mind, or as constituting the material world. But we cannot, or can only with great difficulty, think both things at the same time. To get from the mind-aspect to the matter-aspect or vice versa, we have, as it were, to take the elements asunder and to put them together again in an entirely different order. For example - it is not easy to give examples, but I'll try - my mind at this moment is constituted by all I sense around me: my own body, you all sitting in front of me and very kindly listening to me, the *aide-mémoire* in front of me,

---

1. Schrödinger, *Nature and the Greeks*, [(144)], p.90.
2. *Loc. Cit.*, p.92.

and, above all, the ideas I wish to explain to you, the suitable framing of them into words. But now evisage any one of the material objects around us, for example my arm and hand. As a material object it is composed, not only of my own direct sensations of it, but also of the imagined sensations I would have in turning it round, moving it, looking at it from different angles; in addition it is composed of the perceptions I imagine you to have of it, and also, if you think of it purely scientifically, of all you could verify and would actually find, if you took it and dissected it, to convince yourself of its intrinsic nature and composition. And so on. There is no end to enumerating all the potential percepts and sensations on my and on your side that are included in my speaking of this arm as of an objective feature of the 'real world around us'.

The following simile is not very good, but it is the best I can think of: a child is given an elaborate box of bricks of various sizes and shapes and colours. It can build from them a house, or a tower, or a church, or the Chines wall, etc. But it cannot build two of them at the same time, because it is, at least partly, the same bricks it needs in every case.

This is the reason why I believe it to be true that I actually do cut out my mind when I construct the real world around me. And I am not aware of this cutting out. And then I am very astonished that the scientific picture of the real world around me is very deficient. It gives a lot of factual information, puts all our experience in a magnificently consistent order, but it is ghastly silent about all and sundry that is really near to our heart, that really matters to us. It cannot tell us a word about red and blue, bitter and sweet, physical pain and physical delight; it knows nothing of beautiful and ugly, good or bad, God and eternity. Science sometimes pretends to answer questions in these domains, but the answers are very often so silly that we are not inclined to take them seriously.

So, in brief, we do not belong to this material world that science constructs for us. We are not in it, we are outside. We are only spectators. The reason why we believe that we are in it, that we belong to the picture, is that our bodies are in the picture. Our bodies belong to it. Not only my own body, but those of my friends, also of my dog and cat and horse, and of all the other people and animals. And this is my only means of communicating with them.

Moreover, my body is implied in quite a few of the more interesting changes - movements, etc. - that go on in this material world, and is implied in such a way that I feel myself partly the author of these goings on. But then comes the impasse, this very embarrasing discovery of science, that I am not needed as an author. Within the scientific world-picture all these happenings take care of themselves, they are amply accounted for by direct energetic interplay. Even the human body's movements 'are its own' as Sherrington put it. The scientific world-picture vouchsafes a very complete understanding of all that happens - it makes it just a little too understandable. It allows you to imagine the total display as that of a mechanical clockwork, which for all that science knows could go on just the same as it does, without there being consciousness, will, endeavour, pain and delight and responsibility connected with it - though they actually are.

And the reason for this disconserting situation is just this, that, for the purpose of constructing the picture of the external world, we have used the greatly simplifying device of cutting our own personality out, removing it; hence it is gone, it has evaporated, it is ostensibly not needed. In particular, and most importantly, this is the reason why the scientific world-view contains of itself no ethical values, no aesthetical values, not a word about our own ultimate scope or destination, and no God, if you please. Whence came I, whither go I?

Science cannot tell us a word about why music delights us, of why and how an old song can move us to tears. Science, we believe, can, in principle, describe in full detail all that happens in the latter case in our sensorium and 'motorium' from the moment the waves of compression and dilation reach our ear to the moment when certain glands secrete a salty fluid that emerges from our eyes. But of the feelings of delight and sorrow that accompany the process science is completely ignorant - and therefore reticient.

Science is reticient too, when it is a question of the great Unity - the One of Parmenides - of which we all somehow form part, to which we belong. The most popular name for it in our time is God - with a capital 'G'. Science is, very usually, branded as being atheistic. After what was said, this is not astonishing. If its world-picture does not even contain blue, yellow, bitter, sweet - beauty, delight and sorrow - , if personality is cut out of it by agreement, how should it contain the most sublime idea that presents itself to the human mind?

## 12.2 Equality as Relative Indifference

Equality is cosidered to be context-based, i.e. circumstantial. Equality is always investigated (taken to hold) *relative to some pre-supposed context* which is more or less unspeakable and therefore has to be tacitly agreed upon. For instance, when we state that ‘X is equal to Y’, we obviously disregard the difference in shape of the two letters. In fact, equality always requires *someone*, i.e. an externally observing subject, in order to compare the two candidates. Hence:

**Fact 17:** *Equality* is always taken *relative to an observer* (=*someone*). Two things are *equal* (*equivalent*) when - and only when - the observer chooses to disregard their difference[1].

In this way, each kind of equality is equipped with its own *protocol-of-disregards*, a kind of protocol that defines ‘in what sense’ the corresponding items are ‘equal’ to one another. We could say that we describe the equality in terms of its *disregarded concerns.* This forces us to attempt to formulate these concerns, which places the given type of equality within a more general context and high-lights the equivalence-classes that are involved in its application. This is the essence of *abstract thinking* - being sensitive to what differences and aspects of a problem that can be disregarded without losing touch of its essential structure.[2] Relevance is obtained by systematically disregarding irrelevance, which is expressed in the following well-known

**Fact 18:** The power of thinking is knowing what not to think about.

**Definition 24:** The *disregarded concerns of an equality* are are called its *indifference-space.*

The observing subject was (implicitly) introduced into mathematics by Kurt Gödel in his famous paper of 1931 [(62)]. There he proves the unavoidable existence of certain logical statements within any formal theory that embraces the infinity of natural numbers (in a recursively defined way). These statements are basically just formal encodements of the phrase ‘this statement cannot be proved’, which implies that they are not logically decidable, i.e. ‘verifiable’ or ‘falsifiable’ within the formal system. However, they are by their very nature ‘true’, but *this truth is a form of ‘meta-truth’, which necessarily requires an externally observing subject in order to be recognized.*

From a philosophical point of view, we can summarize these achievementsis in

**Fact 19:** In [(62)] Gödel establishes the logical necessity of *subjectifying mathematics* once and for all. He ‘writes the observer into the equations’, so to say, without any logical possibility of ever ruling her out.[3]

---

1. either because (s)he is unaware of - or uninterested in) this difference.
2. A well chosen abstraction can often disregard almost every aspect of a problem. Classical examples are given by e.g. Newton’s law of gravitation or Maxwells equations for the electromagnetic field.
3. In his book Gödel, Escher, Bach [(76)], Douglas Hofstadter presents some striking illustrations of this point.

# 13 A Formal Model of Participator Consciousness

We make use of our Perception/Conception Duality (yin/yang pair) of Induction/Deduction in order to build coherent patterns (representations) of the world and our own relations to it.

What came first, the experiment or the conceptual framework? This a ‘chicken-and-egg’ type of situation where each one needs the other in order to develop properly. Hence they display a duality which is similar to the yin/yang of ancient Chinese philosophy.

## 13.1 Making Sense is Disregarding Nonsense

Whenever it comes knocking on your windows, experience presents itself as a set of readings on your sense-monitors. Two different knocks will trigger the same reading it the monitor is unaware of their difference. The process of *sensing* is equivalent to *ignoring nonsense*.

**Definition 25:** A *perceptor* is an organ that is aware of the universe.

**Definition 26:** *Awareness* , i.e. *being aware*, is a mapping from the universe to the perceptor.

**Definition 27:** *Im*(Awareness) = *Sence*(Perceptor) is the space of *perceptor-awareness*.

**Definition 28:** *Ker*(Awareness) = *Nonsense*(Perceptor) is the space of *perceptor-ignorance.*

**Fact 20:** *Sense = Universe / Nonsense.*

According to Fact (20), The sensual image is always perceived modulo its corresponding kernel of perceptor-awareness. Hence, a perceptor makes sense from the universe by disregarding its own non-sense. It is the *nature* of perception to experience (= conceptualize) reality as a direct sum of sense and nonsense. It is the *aim* of perception to reconstruct - from its memory of the universe as it *was*, modulated by the sensual image as it *is* - a unique experience which is coherent with the others. It is the *purpose* of perception to reconcile the inductive evidence of the image with the deductive predictions of the conceptual model . It is the *structure* of perception that this reconstruction is performed by a suitable choice of non-sense, guided by deductive assumptions in the model. (“We assume that ..., hence we can conclude that ...”)

$$\boxed{\textbf{Aspect}} \hookrightarrow \boxed{\textbf{Reality}} \twoheadrightarrow \boxed{\textbf{Disregard}}$$

**Fig. 39. The short exact sequence of a disregarded aspect.**

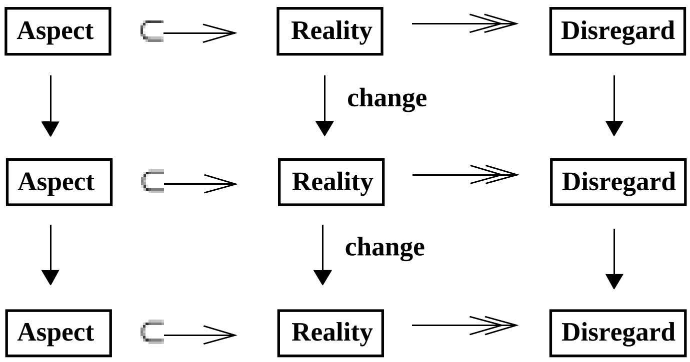


**Fig. 40. The chain complex of disregarded aspects.**

**Definition 29:** Each *change*, the effects of which are neglected by its iterates ($\text{change}^2 = 0$) creates a *chaincomplex*. Each aspect of it creates a *subcomplex*. The disregard of this aspect creates the corresponding *quotientcomplex.*

## 13.2 The Participator as a System of Perceptors

**Definition 30:** A *system* **S** is a family of objects and arrows - with at most one arrrow between each pair - which is *closed* under concatenation, i.e. the concatenation of two subsequent arrows of **S** is alway an arrow of **S**.

A system determines a coherence condition between its different parts. It is clear that if one follows two different paths between to given objects in a system, the resulting arrow will always be the same (since the diagram is commutative). We say that a system is *internally coherent,* or that it is *coherent with itself.*

**Definition 31:** A *participator* is a *system of perceptors*, linked together in some form of *biological unity.*

Each specific *implementation* of such biological unity (= type of species) implies its own specific *coherence condition* on the performance of the individual perceptors in relation to each other. Such a condition can be thought of as a form of ‘common protocol’ for mutual interaction. In fact let us generalize the concept of coherence, by making the following

**Definition 32:** Two systems are said to be *mutually coherent* (= satisfy a *coherence relation*) when they conform to a *common protocol.*

## 13.3 The Actor/Reactor Consciousness

Consider the participator $P$, given by the system of perceptors $\{P_i : i \quad I\}$. We will denote this relationship by $P = \quad P_i$ .

**Definition 33:** The *actor- and reactor consciousness*, $P_{Act}$ respectively $P^{React}$ of the participator $P = \quad P_i$ , are defined by:

$$P_{Act} = \lim_{\rightarrow} P_i \quad , \qquad P^{React} = \lim_{\leftarrow} P_i \tag{7}$$

In the role of *actor*, the coherence protocol is implemented by *'obedience'* (we act coherently == we obey), and in the role of *reactor* the coherence protocol is implemented by *'agreement'* (we react coherently == we agree). Hence the concepts of *obedience and agreement* form a kind of dynamic duality that reflects the corresponding structure of the 'coherence-protocol' of the actor/reactor duality of behaviour. Expanding the semantics, we can write

$$Participator_{Act} = \lim_{\rightarrow} Perceptor_i \quad , \qquad Participator^{React} = \lim_{\leftarrow} Perceptor_i$$

The <*Target*|*Observer*> notation is introduced in [(114)]. Since a participator has both a <*Target*| and an |*Observer*> aspect, we can introduce the natural representation

$$Participator = Target \quad Observer \tag{8}$$

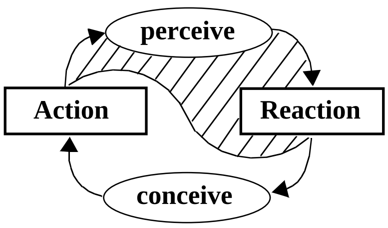


**Fig. 41. The Yin-Yang type of interaction duality.**

This makes the target-observer aspects correspond precisely to the natural projections:

$$\begin{aligned} Participator| &= Target \quad Observer| = Target| \\ |Participator &= |Target \quad Observer = |Observer \end{aligned} \tag{9}$$

From (3) and (4) we can deduce four natural types of participator behaviour:

$$\begin{aligned} &Target^{React} \quad Observer_{Act} \quad , \qquad Target_{Act} \quad Observer^{React} \\ &Target^{React}_{Act} \quad Observer \quad , \qquad Target \quad Observer^{React}_{Act} \end{aligned} \tag{10}$$

Theses behaviours are represented by the four different rounded squares of Figure (42)

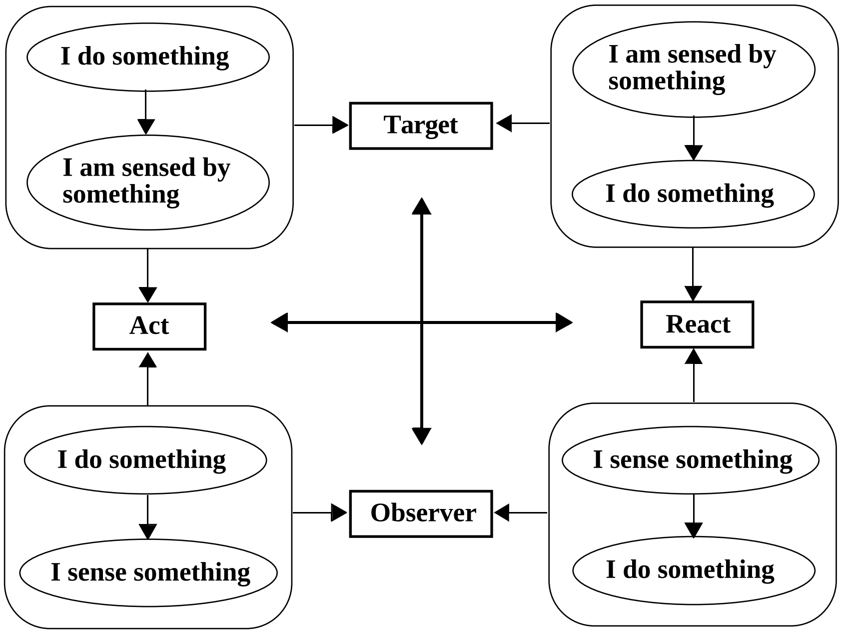


**Fig. 42. The Act-React as a Target-Observer patterns.**

**Definition 34:** Given a participator *P*, a *change* is a couple **U** -> **U** ->*P* -> **U** (before, after). Two changes are *coherent*, if there are morphisms *P*->*P*->*P* such that the diagram commutes.

For each individual participator *P* , this defines a class of coherent changes in a natural way. Moreover, in the spirit of Poincaré[1] I make the following:

**Definition 35:** *Space* is the class of *invertible coherent changes* (= the *undoable*).

**Definition 36:** *Time* is the class of *non-invertible coherent changes* (= the *non-undoable*).

The *concepts* of a participator are the structures that remain invariant under a large enoug sub-group of its actions and its reactions. These are the *nouns* of its conceptor geometry. The *verbs* of the conceptor geometry are the *changes* that are perceptible and coherent to the participator. Each participator has an *anticipator* and a *verificator.*

**Definition 37:** The participator is said to be *relaxed* when its verification is *exact*.

In this case we have the *exact triangle*: *Before->After->Participator* depicted in Figure (43).

1. *Science and Hypothesis*, [(128)].

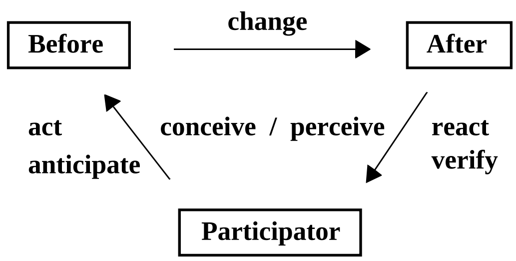


**Fig. 43. The exact triangle pattern of anticipation and verification.**

**Fact 21:** Science contains a non-exact couple of deductive *anticipation* and inductive *verification.* The so called “scientific process” is driven by the lack of exactness for this couple, which goes by the name of *astonishment*, [1]and which is inextricably linked to *curiosity.*

## 13.4 Building a Knowledge Manifold by Calibrating Knowledge Patches

A Kowledge-Manifold represents a kind of network that carries within itself the ‘informational coherence’ of a culture. Any implementation of a Knowledge Manifold is constructed - just as a standard mathematical *manifold* [2]- by the successive *calibration of local patches* (= *overlapping coordinate transformations*).

An important part of a Knowledge Manifold (KM) is given by its *calibration protocols.* This is a set of rules which are either *obeyed* (in the *active* mode) or *agreed upon* (in the *reactive* mode) - depending on how you choose to *interact* with the surrounding patches. By defining an *idea* (= *concept*) as a *representation* of an *experience*, [Definition (21)], a Knowledge Manifold starts from a collection of *subjective* views (individual Knowledge Patches) that are linked together - by *calibration of individual views* - into the *objective world* , i.e. the *‘real world’* around us . This way of conceptual modeling focuses on the *individual* - as opposed to the *collective* - way of relating to knowledge. It is designed as a framework to support *subjective education*, i.e. individually based learning strategies that are developed in a cooperative process between the learners and the teachers.

Figure (44) shows the basic pattern for such calibration activities. Tokens are used as an abstract place-holder for recording the mode of activity. This discussion is connected to CCC-transaction model, presented in Figure (45).

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1. According to Schrödinger [Nature and the Greeks, Cambridge Univ. Press, 1996 (1948), p. 57], ““The first requirement of a scientist is to be curious. He must be capable of being astonished and eager to find out. Plato, Aristotle and Epicurus emphasize the import of being astonished (θαυμαζειν ). And this is not trivial when it refers to general questions about the world as a whole; for, it is given us only once, we have no other one to compare it with.”
2. A *manifold* is a mathematical concept that provides a suitable metaphor for a general (= abstract) description of the knowledge-formation-process. In modern mathematics, a *manifold* is just an *atlas* of patch coordinate systems that *overlap smoothly* with each other. If the atlas is *maximal*, it is called a *differentiable structure,* and we speak about a *differentiable manifold.* This concept plays a fundamental role in the mathematical description system of (= theoretical) physics.

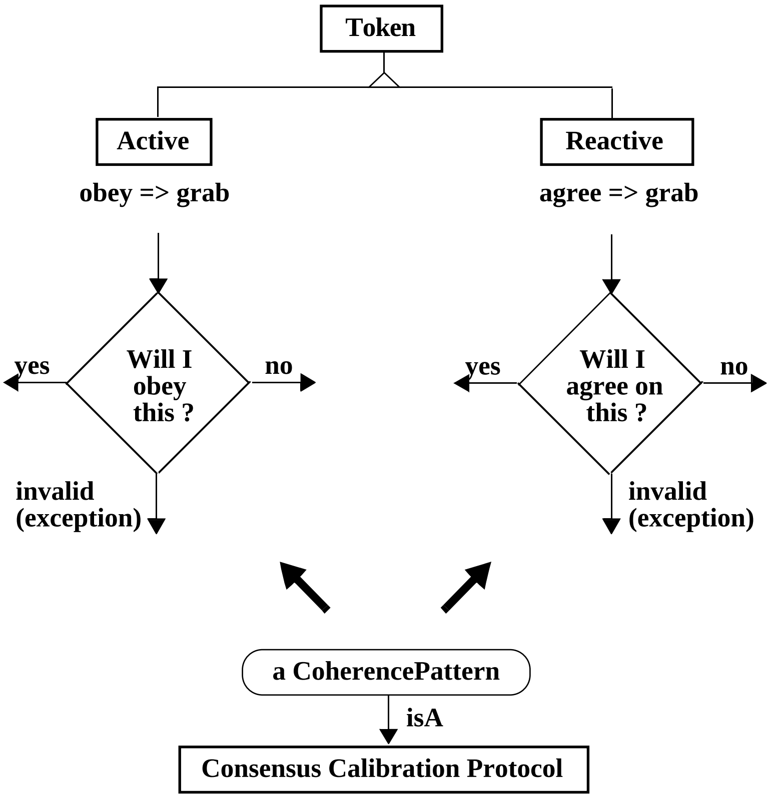


**Fig. 44. The structure of the action and reaction calibration protocols.**

# 14 Future Work

## 14.1 Conceptual Navigation in Structured Information Spaces

In this paper I have presented a mosaik of different ideas centering around the problem of how to create effective educational environments. The metaphor of the Garden of Knowledge as well as the abstract conceptual framework of a Knowledge Manifold were both developed with this overall purpose. During the coming fall there will be process to define the future of the GOK-project. The emerging information technology of cyber-space offers enormous possibilities to support the learning process and enhance it in various ways. In a situation where the traditional school systems across the world are faced with serious difficulties in stimulating interest and curiosity among their students, it is necessary to explore new ways of performing this vitally important task.

I have hopes that the project can continue its exploration of how to present a mixture of theoretical and practical knowledge within various forms of cyber-environments, such as e.g. Dive or Active Worlds. This involves a number of interesting research questions, as e.g. how to utilize the merits of 3-dimensional representations of knowledge-component-graphs, such as the one presented in Figure (38), in order to support the process of navigating in large information spaces. Also, the use of emerging standardized digital informational formats - such as HTML, VRML, MIDI, etc - will be helpful in exploring new possibilities for the design of knowledge-components - in accordance with the principles of Chapter (11.6).

## 14.2 The Virtual Classroom

The *'virtual classroom'* is a generic name for various forms of interactive educational environments. It can be seen as a form of electronic switching-board, facilitating knowledge-transfer across cyber-space. Below I present a sketched outline of a transaction model that incorporates this interactive structure as a special case. It can also be used to model the multi-dimensional money-transactions of electronically supported bartering and trade. The structure is modeled on top of a publically available class-library - called JSDA - written in Java.

### 14.2.1 The Controller-Commodity-Client Transaction Model

The backbone structure of the model is shown in Figure (45). *Commodities* of type *Resource* or *Service* are controlled by *Controllers* of type *Manager* or *Agent*, and *supplied* and *used* by *Clients* of type *Target* and *Observer.* An observer is of type *Consumer* or *Monitor,* depending on whether (s)he is interested in *consuming* data or *monitoring* certain changes or conditions in the data-flow. *Commodities* are supplied by *Targets*, booked by *Agents* and managed by *Managers*, etc. The type hierarchies that emerge are basically reproduced under each CCC-branch, according to the so called *abstract factory* pattern, which is familiar within software design[see e.g. (50)].

A *Session* is a type of *Service* which involves several clients exchanging information. A *Multiparty* is a special type of *Session* where this exchange is carried out over several *Channels* where the clients can have different roles in different channels.

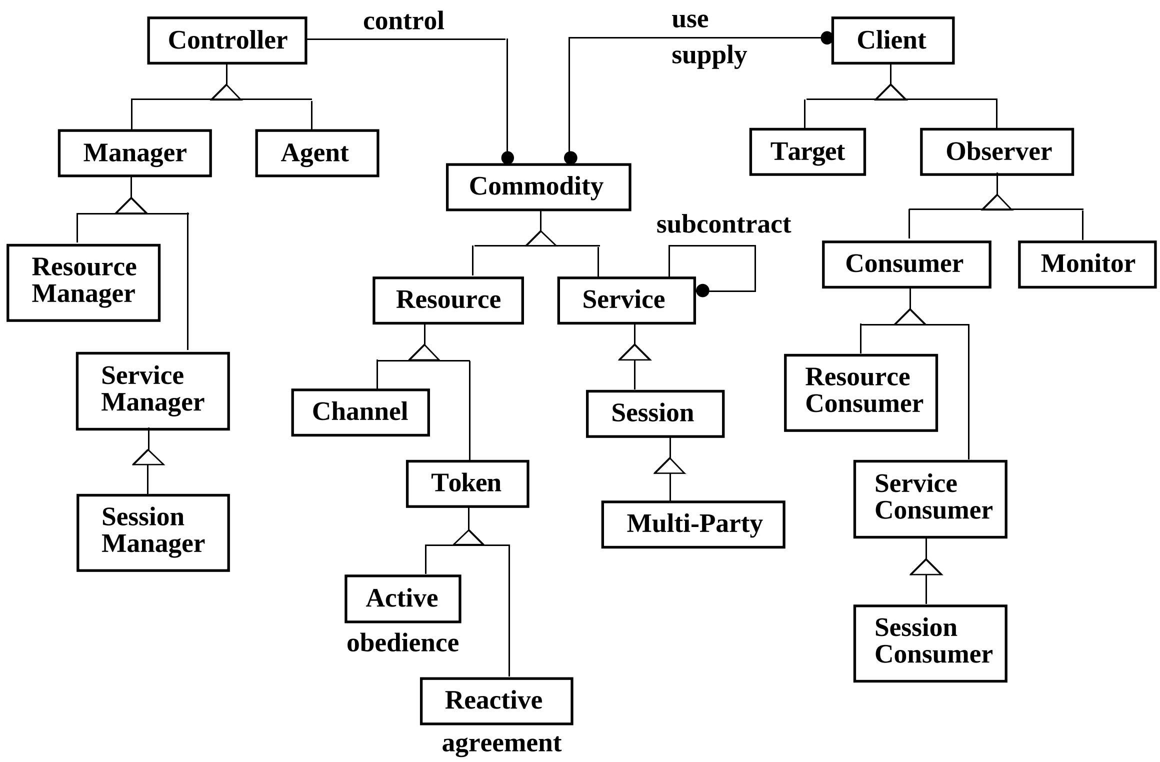


**Fig. 45. The Controller-Commodity-Client model class diagram**

### 14.2.2 Controller-Commodity-Client model : An Outline of a Data Dictionary

What follows is a short, informal description of the required functioning of some of the major objects involved in the CCC-transaction model.

*Client*

An object which is part of application or applet and is a (potential) participant in an instance of multiparty communications. Once properly associated with one another (see Session), related Clients can transfer data in a point-to-point or multipoint fashion. A Client object can be the source or the destination of the data which is being exchanged in an instance of communication. Any number of objects in a Java applet/application can be defined to be Client objects (with respect to this multiparty communications API).

*Session*

A collection of related Clients which can exchange messages via defined communications paths (see Channel ). The Session maintains the state associated with the collection of clients and their associated communications paths, and may interact with an object which encapsulates a defined session management policy (see Manager ). The process of negotiating for, and selecting, the specific transport protocol to be used is done transparently at Session instantiation. An application or applet can have multiple Client objects associated with the same (or different) Session objects.

*Session* (dynamically)

This service establishes a multipoint Session over point-to-point connections. Within that multipoint Session a Client can send data to different members of the Session and have access to Tokens for resource contention resolution.

The first thing a Client has to do is join a Session, using the *join* Session method. The Session will typically have multiple Clients (either at the same site or at other sites). An application or applet can have multiple Clients in the same Session. Each Client might be handling a different kind of data (ie. audio vs video). A Client can be a member of multiple Sessions.

The *leave* Session method is used by a Client to leave a Session. Once the Session is established, the Client then joins the appropriate Channels it requires for receiving data. The use of these Channels is application dependent. The Client should then let the Channel know who is going to be the Channel consumers of data sent over this Channel. Tokens are also provided for managing the resources available to the clients.

*Channel*

A specific instance of a, potentially multi-party, communications path between two or more Clients within a given Session. All Client objects which register an interest in receiving from a given Channel will given messages sent on that Channel (see Consumer below). Any Client which possesses an object reference to a Channel is able to send a message on the given Channel, and a Client can have references to multiple different Channels.

*Channel* (dynamically)

Once the Session setup and Client attachment is completed, the last step to be performed before data can be exchanged between all the members in a multipoint fashion, is to join the right combination of interaction channels. Channels are session-wide addresses. Every Client of a session can join a Channel to receive data sent to it, and by joining an appropriate combination of Channels, and by consuming them, a Client can choose to receive Messages sent to these Channels and ignore Messages sent to other Channels. Clients subscribe to and leave the desired Channels with the join and leave Channel methods.

*Consumer*

A Client object which has registered its interest in receiving messages sent over a given Channel. Any Client can be a Consumer, and it is possible for a given Client object to be a Consumer of multiple Channels a the same time.

*Consumer* (dynamically)

A Client sets a Consumer on a Channel to receive Data sent over that Channel. The Data contains the raw data, plus the senders name, the priority the Data was sent at and the Channel the Data was sent over.

*Monitor*

A Client object that has registered its interest in being notified about changes in state of some other given Client object.

*Monitor* (dynamically)

A Monitor can observe changes in a Session, on a Channel or with a Token. A *Session Monitor* will be notified about Clients joining and leaving a Session. A *Channel Monitor* will be notified about Clients joining and leaving a Channel, plus a Client being invited to join a Channel or expelled from a Channel. A *Token Monitor* will be notified about Clients joining and leaving a Token, plus a Client being invited to join a Token or expelled from aToken. They are also notified when a Client wishes to give or take a Token from another Client.

*Manager*

A Controller-type of object which encapsulates some management policy for another given object. For example, a Manager is used in order to authenticate Clients to determine whether they are allowed to join a given Session.

*Manager* (dynamically)

Access to a Session, Channel or Token can be controlled by assigning a Manager to it at creation time. The Manager is responsible for authenticating Clients wishing to join this Commodity, and based upon their response with accept ot reject them. Both send and receive access to individual channels can be controlled by a private channel mechanism. Any Client may convene a private Channel with the convene Channel method, which results in him/her becoming the private channel manager of an empty channel. The *Channel Manager* can invite a Client to join the Channel using the invite Channel method or force a Client to leave with the expel Channel method. Clients join and leave these private channels using the regular join and leave Channel methods.

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The same process occurs for Tokens using the equivalent Token methods.

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*Token*

A synchronization type of Resource object which provides a unique distributed atomic operation. Tokens can be used to implement a variety of different application-level synchronization mechanisms (e.g., to ensure mutually exclusive access to a shared Resource).

*Token* (dynamically)

Tokens provide a means to implement exclusive access. For example, to ensure in a multipoint application using resources that one and only one site holds a given Resource at a given time, a

Token can be associated with every Resource. When a site wishes to use a specific Resource, it must ask for its corresponding Token, which will be granted only if no one else is holding it. The *grab Token* method allows one Client to exclusively hold a given Token. The Client defines the significance of this Token in the application. Other Clients may use the *test Token* method to determine the status at any time and may request the Token from the holder with the *please Token* method. The token holder may transfer control of a Token to another specified Client with the *give Token* method or return a Token to a generally available status with the *release Token* method. A single token may be used to coordinate a multiple client event by using the grab Token method in a non-exclusive mode. Clients can independently inhibit and release the same token. For example, if it was desired to know when all clients have completed reception and processing of a bulk file transfer, all receiving clients would non-exclusively grab (inhibit) the same token and each individual client would release the token when it had completed the proscribed process. Any client could test the token at will to determine if the token is free which means all the clients have completed processing.

*Data*

Once the participating Clients have attached to the common Session and joined the right combination of Channels, they are ready to exchange Data in a multipoint fashion. The *send* and *uniformSend* Channel methods provide the actual transfer of data. Each send data unit can be delivered to multiple sites. Uniform data sequencing, or delivery at each site of identical sequences of Data, is provided with the uniformSend Data method. The *send* Data method provides the "one-to-many" communication, which includes point-point as a particular case. Simple send data from different senders may arrive in a different sequence at each site, since they are delivered using the most direct route. Some sequencing occurs even without using the uniformSend Channel method. The sequencing of data sent from one sender on one Channel at one priority is maintained identically at all receivers. The *uniformSend* Channel method offers this service. All the uniformSend requests are dispatched in the same order to all the receiving sites.

(Note: Uniform sequencing is necessary when data is being sent simultaneously from several sites, but must be received in the same sequence by all receivers.)

*Message*

A discrete unit of data that is sent by a Client over a Channel to all of the Clients which have registered an interest in the given Channel.

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