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On Simultaneous Online Auctions with Partial Substitutes

Vipul Bansal, Rahul Garg
email: {bvipul,grahul}@in.ibm.com

IBM Research Division,
IBM India Research Lab,
Hauz Khas, New Delhi - 110016. INDIA.
Phone: +91-11-6861100, Fax: +91-11-6861555

On Simultaneous Online Auctions with Partial Substitutes

Vipul Bansal, Rahul Garg
{bvipul, grahul}@in.ibm.com
IBM India Research Lab.,
Hauz Khas, New Delhi - 110016, INDIA.

Abstract

Millions of online electronic auctions on the Internet today are being conducted as isolated auctions, leading to inefficient outcomes. We argue that if the independent ascending auctions of substitutes (similar items) are made simultaneous and agents are provided to the customers to bid across a number of these auctions, then the resulting system has several desirable properties. In particular, we show under very general conditions, that if the agents follow a simple truth telling greedy bidding strategy (which we call LGB), then a Nash equilibrium results which not only maximizes the total surplus of the system but also distributes it fairly among the bidders and the sellers through nearly unique competitive prices. We discuss the implications of our results for the design of online auctions on the Internet.

1 Introduction

Commerce is about transfer of goods and services to entities who value them more from entities who value them less, thus creating value (surplus). There is a justification for electronic commerce if it creates additional value, either from cutting costs or doing better allocation.

Commerce on Internet is poised to constitute a significant fraction of the entire global commerce in the years to come. Conventional models like fixed price sale, auctions and double auctions are now being used to buy and sell on the Internet. Auctions fall somewhere in between the fixed price and double auction mechanisms - they fulfill the need to sell (or buy) unique items where there is a single seller and multiple potential buyers. Such items cannot be sold by double auctions as the items are not standardized. They cannot be sold through fixed price sales because the demand and 'true' prices of such items are often not known and need to be discovered in the process. The reader is referred to [1, 2] for an introduction to auctions.

In general, the outcome of an auction defines the allocation (which items go to which bidders) and the prices. The allocation determines the surplus while the prices determine the sharing of the surplus between the auctioneer and the bidders. An efficient allocation is one that maximizes the global total surplus in a given setting [3]. In case of a single item, certain auctions under some general conditions, are known to result in efficient allocation and fair and competitive pricing [4].

An auction outcome is the result of the interaction of bidding strategies adopted by the bidders. A profile of strategies is said to constitute a Nash equilibrium [5] if given that all bidders adhere to the prescribed strategies, no bidder can gain by unilaterally switching to another strategy. A strategy is dominant if it maximizes a bidder's payoff independently of the strategies of the other bidders. Therefore, if a mechanism has a dominant strategy, the bidders would choose the dominant strategy above all other strategies.

The conventional open cry auction is designed for a physical setting where a bidder can participate in only one auction at a time. Even though this restriction is no longer relevant to the auctions on Internet, the auction sites continue to use the *isolated auctions* model. A better model would take into account the presence of substitutes being sold simultaneously and the presence of all competing bidders in determining the allocation of items and their prices.

The present work proposes a way of conducting simultaneous online ascending auctions of substitutes. The proposed mechanism allows sellers to auction the items independently using open cry auctions, but require that the auctions open and close at the same times. If this condition is met, we show that a simple bidding strategy exists (which software bidding agents can execute) which maximizes the surplus in the system and redistributes it fairly amongst the participants through nearly unique competitive prices. The strategy constitutes a Nash Equilibrium for the system under very general conditions and is also a dominant strategy under somewhat restrictive conditions. It also has many other properties desirable from a practical point of view.

Shapley and Shubik [6] considered an assignment market where each participant wants one item and places a monetary value on each item in the market. A participant demands the item which, given a price vector for the items, maximizes its surplus. It was shown that if the valuations of all participants for all items are known then the model always has an equilibrium and there is a unique smallest price vector P such that P is at least as small in every component as any other equilibrium price vector. Leonard [7] and Demange and Gale [8] showed that a sealed bid multi-item auction of substitutes, where each bidder wants one item, is incentive compatible if the smallest equilibrium price vector is used as the price setting rule. This implies that submitting their true valuations is the dominant strategy for bidders. This allowed the results derived by Shapley and Shubik [6] to be directly applicable to multi-item sealed bid auction of substitutes. This also established the multi-item sealed bid auction as a generalization of the second price sealed bid auction of a single item [4].

Bikhchandani and Mamer [9] extended the results of Shapley and Shubik [6] to an exchange economy of indivisible items where a participant may want a subset of items instead of a single item. The participants' preferences are specified by non-decreasing monetary values for each subset of items, i.e., if $T \subset S$ then $value(T) \leq value(S)$. They derived the necessary and sufficient conditions for the existence of market clearing prices and showed that if the market clearing prices existed, the corresponding allocation would be efficient.

Do there exist market mechanisms, in particular, which can be implemented in the form of online mechanisms on the Internet, which can efficiently allocate multiple items?

Milgrom [10] examined simultaneous ascending auctions for substitutes in the context of FCC auctions [11]. He considered a multi-round sealed bid auction for multiple items and showed that if the bidders bid 'straightforwardly' (place bids in the next round for the subset of items that maximizes their current surplus), then a competitive equilibrium results which maximizes the surplus in the system up to a number proportional to the bid increment. However, the straightforward bidding strategy is restrictive since it requires that all bidders be present right from the beginning of the auction and place a bid in each round (i.e., they cannot temporarily withhold their demands). In practical online auctions, it should be possible for bidders to join any time and stay inactive (or withhold demand) for some part of the auction.

Demange et. al. [12] considered an ascending auction where the bidders want only a single item out of a set of heterogeneous items. They showed that *if* the bidders announced honestly at each stage the item whose value to the bidder exceeds its current price by the maximum amount, the auction mechanism nearly converges to the smallest equilibrium price vector P with each component of P being within $m\epsilon$ (where m is the number of items sold and ϵ is a minimum required bid increment) of the corresponding smallest equilibrium price. Miyake [13] showed that under assumptions of monotonicity and continuity on the bidders' preferences, the honest strategy assumed by Demange et. al. [12] converges to a dominant strategy in the limit of the bid increment approaching zero. Wellman et. al. [14] further showed that the inefficiency in the resulting allocation is bounded by at most $m(m+1)\epsilon$.

While the auction mechanism suggested by Demange [12] et. al. is easily implemented for online auctions on Internet in the form of independent simultaneous ascending auctions of items, many problems remain. The bound on inefficiency obtained by Wellman et. al. [14] does not augur well for such auctions because it seems to suggest that the inefficiency per item (in a system of m items on simultaneous auctions) is proportional to m. On Internet, m may be several thousands, and hence the system of auctions does not scale well. We derive a much tighter bound on inefficiency in this paper which removes this problem.

Further, the assumptions on bidder behaviour used by Miyake are restrictive and may not represent all possibilities. It is not clear whether the honest bidding strategy continues to be a dominant strategy or even constitute a Nash Equilibrium in the presence of deviant bidders who do not adhere to the assumptions used by Miyake. In this paper, we show that even under very general conditions, the honest bidding strategy assumed by Demange [12] constitutes a Nash Equilibrium for the system. However, under these general conditions, we show that the strategy ceases to be a dominant strategy.

The rest of the paper is organized as follows: Section 2 formally defines the system of simultaneous independent open cry auctions (SIA). Section 3 defines a truth-telling local greedy bidding (LGB) strategy. In Section 4 we discuss the characteristics of the equilibria of SIA under LGB bidding. Section 5 discusses the implications of our work for online auctions. Finally Section 6 lists the conclusions. Important proofs are given in the Appendix.

2 Simultaneous Independent On-Line Auctions

Auctions are simultaneous if they open and close at the same times. We consider simultaneous auctions where each auction is conducted independently as an open cry auction. In each of these auctions, bidders may place their bids at any time on any item so as to outbid the current highest bid on that item. For each item, the bidder with the highest bid at the close of the auction is declared the winner and pays an amount equal to its bid to get the item. To ensure that the auctions terminate in a finite number of steps, a new bid must exceed an existing bid by a minimum increment (ϵ). If bidders are software agents, ϵ can be set to a very small value.

We consider a set A of m items auctioned simultaneously in open cry manner. The seller of each item j sets a reserve price p_j^r, equal to its valuation for the item. Let B be the set of n bidders in the system. Each bidder i has a preference set of items, $A_i \subset A$. For each item $j \in A_i$, the bidder i has an independent private valuation v_{ij}. The bidder i wants at most one item from amongst the items in A_i. Therefore, the bidder i regards the items in A_i as substitutes. They are however, only partial substitutes in the sense that the bidder i has different valuations for the different items in A_i.

Let P^r represent the m x 1 vector of reserve prices p_j^r and let V represent the n x m matrix of valuations v_{ij}. The system of simultaneous independent open cry auctions described above is now represented as $SIA(A, B, V, P^r)$. The status of the auction at any time can be stated in terms of the vector of current prices (winning bids) for the m items, P.

In a more general setting, there are multiple identical copies of each item on auction. For all q_j copies of an item j, a bidder i is assumed to have identical valuations v_{ij} and the auctioneer sets the same reserve price p_j^r for them. The multiple copy scenario is easily mapped to the single copy scenario by treating each copy of an item as a separate item. Set A now contains $\sum_{j=1}^{m} q_j$ items instead of m items. It is easy to see that since each bid in the system is for only one unit, and since the bidders regard the items as substitutes, it does not matter whether the multiple copies of an item are auctioned by the same or different auctioneers. Therefore, we shall consider only the case where each item has only one copy although the results derived will be applicable for the case of multiple copies as well.

Consider that an item j is sold to a bidder i at a price p_j. The seller of j derives a surplus of $p_j - p_j^r$ while the buyer i derives a surplus (s_i) of $v_{ij} - p_j$. The total surplus created by this auction outcome is the sum of the seller and buyer surpluses and equals $v_{ij} - p_j^r$. The total surplus is independent of the price and depends only on the assignment of the items to bidders. The global total surplus for $SIA(A, B, V, P^r)$ is simply the sum of the total surplus created in each of the m auctions. Thus, if the item j was sold to its winner $w(j)$, then the global total surplus is given by: $S_G = \sum_{j=1}^{m}(v_{w(j)j} - p_j^r)$.

3 Local Greedy Bidding

Consider a bidder i who wants to get at most one item out of the m items in $SIA(A, B, V, P^r)$. The current local surplus of bidder i on an item j is defined as $v_{ij} - p_j$ where p_j is the current price of item j. For a bidder i who does not have any winning bid, the item of greedy choice, $g_P(i)$, given the current price vector P is defined as the item with the maximum current local surplus. Thus, $g_P(i) = j$ such that $\forall j', v_{ij} - p_j \geq v_{ij'} - p_{j'}$ and $v_{ij} - p_j \geq \epsilon$. If there exist more than one such item, $g_P(i)$ can be defined as the item with lowest index j without losing generality.

According to the local greedy bidding strategy (LGB) a bidder never has more than one outstanding (winning) bid at any point of time. If a bidder is currently not winning on any item, it determines its item of greedy choice and places a bid of value ϵ greater than its current price. In case there is no such item, the bidder stops bidding. When a bidder is outbid, it uses LGB to determine the next bid.

Note that LGB has the property that at any time during the auction, bidder i's outstanding bid is always on an item such that i cannot obtain a greater surplus by placing a bid on any other item. By definition of LGB, this is true at the time when i places a new bid. From then onwards and until i is outbid, the prices of other items can only rise (and so i's surplus on them can only fall).

We assume that there are m dummy bidders in the system, one bidder corresponding to each item in A. The dummy bidder corresponding to the item j has a valuation of p_j^r for j and zero for all other items. The set of bidders B is assumed to include these dummy bidders. At the start of the auction, these dummy bidders place their respective bids at reserve prices on the respective items prior to any bids being placed by any exogenous bidders.

4 Equilibria with Local Greedy Bidding

Consider the system $SIA(A, B, V, P^r)$ with the bidders following LGB. Define competition on an item j, given the current price vector P for the system, as one (current winner) plus the number of non-winning bidders for whom the item j is the item of their greedy choice. More precisely, $COM_P(j) = 1 + \mid \{i | g_P(i) = j\} \mid$. The competition in the system $SIA(A, B, V, P^r)$, given the price vector P, is defined to be the sum of competitions on all the m items on auction, $COM_P = \sum_{j=1}^{m} COM_P(j)$. At the start of the auctions $P = P^r$ and for all j, $COM_P(j) \geq 1$ due to the presence of dummy bidders.

Every new bid either decreases COM_P, or leaves it unaffected. A bid on an item j causes its price p_j to increase by ϵ. This can cause a bidder who was competing on j to either (a) place a new bid on the new item of its greedy choice, or (b) quit the system if no such item exists. In the former case COM_P remains unchanged while in the latter, it falls. The bidders not competing on j are not directly affected by the new bid on j.

Lemma 1 (Termination) *The bidding in system $SIA(A, B, V, P^r)$, with the bidders following the local greedy bidding strategy terminates in a finite number of steps. When bidding terminates, $COM_P(j) = 1, \forall j \in \{1 \ldots m\}$.*

The proof is provided in the Appendix. We now have an intuitive picture of how LGB works. When the auctions commence, there are many items on which more than one bidders may maximize their local surpluses. As a result of this competition on a given item, the price of that item increases, making the surplus on it less than the surplus on some other item for some of the bidders who were competing for this item. This causes them to move away to other items (or quit, if there is no item left which gives them a non-negative surplus). Thus, LGB shifts competition from one item to another as bidders keep hunting for items that maximize their local surplus. In the process the prices rise, leading to exit of some bidders, thereby reducing competition in the system.

Theorem 1 (Efficient Allocation) *If S_{OPT} is the maximum global total surplus for the system $SIA(A, B, V, P^r)$ under any allocation, and S_{LGB} is the global total surplus with LGB bidding then, $S_{OPT} - S_{LGB} \leq m\epsilon$.*

The proof is provided in the Appendix. Theorem 1 implies that if ϵ is small, $SIA(A, B, V, P^r)$ with LGB bidding leads to efficient allocation. This theorem implies that the average inefficiency per item for a system having m items on simultaneous auctions is of the order of ϵ, no matter how large m is.

Since the reserve prices are fixed, the allocation is such that the sum of valuations of the winners for their respective items, $\sum_{j=1}^{m} V_{w(j)j}$, is close to the maximum such sum possible for any allocation feasible for the system $SIA(A, B, V, P^r)$. This can be regarded as a generalization from a simple open cry auction of a single item, where the item is allocated to the bidder with highest valuation.

Consider any outcome of $SIA(A, B, V, P^r)$ with LGB bidding. Define *marginal losers* of a subset $C \subset A$ of items as bidders who placed the second highest bid on an item in C and eventually did not get any item from C.

Lemma 2 (Existence of Marginal Loser) *In every outcome of $SIA(A, B, V, P^r)$ with LGB bidding, for any $C \subset A$, either there exists a marginal loser or all the items in C were sold to the corresponding dummy bidders at their reserve prices.*

The proof is provided in the Appendix. The lemma states that if an item was sold to an exogenous bidder, there exists a loser who just lost out on the margin. It signifies that it is not possible for all prices to fall simultaneously by more than ϵ from one outcome to another. If that were to happen, a marginal loser would step in and place a bid. The marginal losers thus hold the final prices. The lemma can be seen as a generalization of simple open cry auction, where the second highest bidder eventually loses but determines the price.

Theorem 2 (Almost Unique Prices) *Consider the system $SIA(A, B, V, P^r)$ with LGB bidding. Let P^1 and P^2 be the final price vectors for two different outcomes of the system. Then the following holds: $\forall j \in A, | p_j^1 - p_j^2 | \leq 2m\epsilon$.*

This result follows from earlier work by Demange et. al. [12]. A bound of $3m\epsilon$ on the price differences is also deducible from Lemma 3 below.

Corollary 1 (Almost Unique Auctioneer Surpluses) *The surplus of every auctioneer is uniquely determined within $2m\epsilon$.*

Lemma 3 (Deviant Bidder) *Consider any outcome of $SIA(A, B, V, P^r)$ with LGB bidding with final price vector P^1. Consider another outcome where all bidders except for a bidder l follow LGB. If bidder l places a bid p_j on item j such that $p_j^1 - p_j > 3m\epsilon$, then it will be outbid.*

The proof is provided in the Appendix. The lemma establishes that no bidder has the ability to obtain lower prices by unilaterally following some other strategy when all others follow LGB. We are able to show this without any assumption on the strategy followed by the deviant bidder.

Theorem 3 (Nash Equilibrium) *Consider any outcome of $SIA(A, B, V, P^r)$ with LGB bidding and let s_i be the surplus of bidder i. Bidder i cannot obtain a surplus greater than $s_i + 3m\epsilon$ by unilaterally following some other strategy. Therefore the system approaches Nash equilibrium in the limit when bid increment approaches zero.*

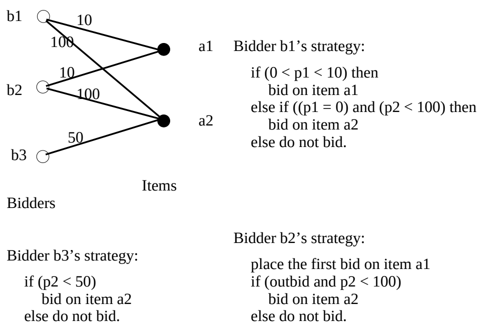

Figure 1: An example where LGB is not a dominant strategy

The proof is provided in the Appendix. Note that LGB constitutes a Nash Equilibrium for the system under very general conditions since no assumptions have been made on what strategy a deviant bidder can adopt for obtaining a greater surplus.

Corollary 2 (Almost Unique Bidder Surpluses) *The surplus of every bidder, when all bidders follow LGB, is uniquely determined within* $3m\epsilon$.

We note that if ϵ is close to zero, the prices and surpluses are unique. It follows that the order in which bids arrive is inconsequential if ϵ is small. We can increase m as long as ϵ can be correspondingly reduced to keep the price uncertainty within acceptable levels. Increasing m implies better allocation (maximization of surplus for a larger system) but it also leads to a greater uncertainty in prices and individual surpluses.

The source of this uncertainty lies in the formation of ties and the order in which bidders may place their bids. If every bidder had an ordering for selecting its item of greedy choice, $g_P(i)$, from amongst the items where it has equal current local surplus, and if the bidders placed their bids in a round-robin manner, then the entire sequence of bids can be determined by a deterministic algorithm, thereby implying a unique allocation and price vector.

Remark 1: Dominant Strategy - A Discussion

For LGB to be a dominant strategy a bidder should be able maximize its payoff by using LGB independent of the strategies used by other bidders. In a general setup where no restrictions are placed on the strategies that the other bidders can use, LGB may not be a dominant strategy.

To illustrate this, consider a system $SIA(A, B, V, P^r)$ as shown in Figure 1 with $A = (a_1, a_2)$, $B = (b_1, b_2, b_3)$, $V(b_1) = (10, 100)$, $V(b_2) = (10, 100)$, $V(b_3) = (0, 50)$ and $P^r = (0, 0)$. Assume that minimum required bid increment, $\epsilon = 1$. Bidder b_1 follows the strategy: if $0 < p_1 < 10$ then bid on a_1 else if $p_1 = 0$ and $p_2 < 100$ then bid on a_2 else do not bid. Bidder b_2 follows the strategy: place its first bid on a_1. On being outbid, if $p_2 < 100$ place a bid on a_2. Bidder b_3 uses the strategy: if $p_2 < 50$, place bid on a_2.

With these set of strategies, b_2 will either get a_1 at a price of 1 (with a surplus of 9) or a_2 at a price of 51 (surplus of 49). However, if b_2 were to follow LGB strategy, it would get item a_2 at a price of either 90 or 91 (surpluses of 10 and 9 respectively) or item a_1 at a price of 1 (and a surplus of 9). Therefore, b_2 may realize significantly greater surplus by following a strategy other than LGB when other bidders follow arbitrary strategies as indicated. Therefore, LGB is not a dominant strategy for the system under general conditions.

However, if the set of strategies that the bidders can follow is suitably restricted, LGB can be shown to be a dominant strategy. Miyake [13] restricts this set to strategies which can be expressed by (a) defining a total order on the set of pairs (item, price) which satisfies the properties of monotonicity and continuity, and (b) bidding at each stage on the item of highest rank given the current price vector. He shows that under these conditions, the truth telling bidding strategy in which the total order is defined by the bidder surplus is a dominant strategy.

Remark 2: Price Determination and Decomposition of SIA

Consider an outcome of LGB bidding and assume that the bid increment is sufficiently small and can be ignored. Suppose bidder i_1 got item j_1 in this outcome. Suppose that bidder i_2 had the second last bid on item j_1 and it got item j_2 in the outcome. Continue to extend the chain in this way until it reached a bidder i_{k+1} who did not get any item. The prices of items in such a chain are related by the following equations:

$$\begin{aligned} p^*_{j_k} &= v_{i_{k+1}} \\ p^*_{j_l} &= p^*_{j_{l+1}} + v_{i_{l+1}j_l} - v_{i_{l+1}j_{l+1}} \end{aligned}$$

The marginal loser i_{k+1} holds the price of all the other items in the chain. A small δ increase in his valuation will increase the price of all the items in the chain by δ. None of the other losers has any impact on the system. This is a generalization of the $M+1^{st}$ price auction of M identical items where the price of all the items is determined by the $M+1^{st}$ highest bid.

For such a chain, the prices of items $j_1 \dots j_k$ are completely determined by the valuations of bidders $i_1 \dots i_{k+1}$. The system $SIA(A', B', V', P^{r'})$ where $B' = \{i_1, \dots, i_{k+1}\}$ and $A' = \{j_1, \dots, j_k\}$ and V' and $P^{r'}$ are appropriately defined, with LGB bidding, converges to nearly the same prices (and therefore the same bidder and auctioneer surpluses) as in the outcome of $SIA(A, B, V, P^r)$ with LGB bidding. Therefore, the system of simultaneous independent auctions can be decomposed into multiple (potentially overlapping) chains, with each chain containing one more bidder (the marginal loser) than the number of items in it.

Remark 3: Collusion and Price Reduction

Consider any LGB allocation. If bidders were to first choose the items which they would respectively win using LGB, and then bid only on them, the resulting prices will, in general be lower than the prices when they use LGB. This is because, when bidders use LGB, some of the second highest bids can come from amongst the winners of some other items. If each bidder chose and bid only for its item, these second highest bids which came from winners (when LGB was used) would be replaced by second highest bids coming from some bidders who did not win any item and also did not have any second highest bids when LGB was used. With LGB, all the prices, exceeded the valuations of these bidders by at least ϵ. But with the new strategy, there may be some prices, which are equal to their valuations or no more than ϵ greater than their valuations.

Therefore, if bidders were to collude (or know each others valuations) then they can follow a strategy of bidding only on their respective items and get them at lower prices. In practice, since the valuations are not known to one another, they will have to compete with each other and discover the allocation and prices.

5 Implications for Online Auctions

We saw that if partial substitutes are sold using simultaneous independent ascending auctions, then a simple and truth telling greedy bidding strategy (namely LGB) results in an equilibrium which has many desirable properties.

The isolated auctions model tries to maximize the surplus for each auction separately and in the process leads to a system outcome which may be inefficient. We saw that LGB bidding in SIA leads to efficient allocations. The additional surplus in SIA with LGB bidding comes from the ability to take into account the valuations of all bidders for all items in deciding the allocation. In the isolated auctions model, the bidders select the items to bid on and given the bidders' selections, the auctions then discover the prices and winners. In SIA with LGB bidding, both allocation and prices are discovered by the mechanism itself, thus avoiding sub-optimal choices.

The system SIA with LGB bidding is also a Nash equilibrium with nearly unique prices. Therefore, given the items and bidders, no auctioneer can increase its surplus by selling its item at a higher price and no single bidder can get a higher surplus by following any other strategy. The resulting allocation of surplus between auctioneers and bidders is competitive and hence fair.

For the bidders, ability to bid across multiple simultaneous auctions results in greater choice. Further, LGB ensures that a if bidder has maximum valuation amongst all bidders for some item, then it will win at least one item. More than that, LGB ensures that the bidder would get the 'right' item that would give it the maximum possible surplus, given the items and the competition. For the auctioneers, simultaneous auctions with bidders bidding across them implies more competition for their items. For the system as a whole, it implies participation by all bidders in auctions of all items, thereby yielding the surplus maximizing allocation of items to bidders.

A very desirable property of SIA is that each auctioneer can conduct its own auction independently as a simple open cry auction. The only requirement is that the auctions be simultaneous. To achieve this, various auction sites only need to group auctions of similar items and agree on common start and end times. Another desirable property is that LGB is a fairly simple strategy and needs only the valuations of a bidder and the current price vector of auctions for bidding. Therefore, very simple software agents can implement this strategy and execute it for bidding across multiple auctions on an auction site and also across multiple auction sites.

Therefore, we argue that the auction sites should redesign their auctions so that the auctions of similar items (potential substitutes) simultaneous. Further, they should provide LGB capable agents for bidding.

6 Conclusions

In this paper, we formally defined the system of simultaneous independent open cry auctions (SIA) with partial substitutes and proposed a bidding strategy called local greedy bidding (LGB). With the independent private valuation assumption, we showed that the LGB bidding results in a nearly efficient allocation for SIA. We also showed that LGB constitutes a Nash Equilibrium for the system under very general conditions. Therefore, no bidder can improve its surplus by unilaterally following any other strategy. Under somewhat more restrictive conditions, LGB is also a dominant strategy for the system.

Bidding in LGB is based on true valuations of items, therefore the system also has good incentive properties. The prices of items sold in SIA under

LGB bidding are nearly unique, competitive and depend only on the bidder's valuations (demand) and the items (supply). We also argued that the surplus generated is distributed fairly among the bidders and the sellers. We identified the bidders who are necessary for the equilibrium to be achieved and in the process identified marginal losers (bidders who placed a second highest bid on some item but did not get any item) who along with the winners determine the prices at equilibrium.

LGB bidding does not require any information other than bidder's valuations and current winning bids. Therefore it should be very simple to provide LGB bidding agents to bidders. The simultaneous auctions need not be conducted by the same auctioneer. Independent auctioneers just need to agree on opening and closing time for auctions of similar items.

SIA with LGB bidding may be seen as a generalization of multi-unit ascending auctions to independent ascending auctions of heterogeneous items with the additional condition that a bidder has *one out of k* demand and may bid in multiple auctions. The foregoing results can be extended further for the case where a bidder wants to get more than one item by suitably defining LGB for the multi-demand case. We have not considered the general case here for simplicity of exposition.

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8 Appendix

Proof: [Lemma 1] Each new bid increases some price in the system by ϵ. Since the valuations in V are finite, and since the prices of items on auction cannot go beyond the valuations, the bidding terminates in a finite number of steps.

Initially, because of dummy bidders, each item has a winning bid. Open-cry auctions ensure that a winning bid gets invalidated only by another winning bid. Therefore $COM_P(i)$ can never go below 1.

Assume for contradiction that at termination, there is an item j such that $COM_P(j) > 1$. Since there can be at most one winning bid on item j, there is a bidder i who is not currently winning on any item and $g_P(i) = j$. In the next step, bidder i should outbid the current winner of item j by placing an LGB bid on item j. Therefore the auction has not yet reached termination, which contradicts our assumption. Hence, at termination $\forall j, COM_P(j) = 1$. □

Proof: [Theorem 1] Let $G = (A, B, E, W)$ be the bipartite graph where A is the set of m vertices representing the m items, B is the set of n vertices representing the n bidders. The edge $(a_i, b_j) \in E$ iff $v_{ij} \geq p_j^r$ and the weight of the edge $W(a_i, b_j)$ is equal to $v_{ij} - p_j^r$. A matching M on G is defined as a set of edges, $M \subset E$ such that for each vertices in A or in B, there is at most one edge incident on it. Any allocation of items to bidders can now be represented as a matching M on G. The global total surplus for any allocation is equal to the weight $W(M)$ of the corresponding matching. An allocation maximizing the global total surplus of the system is also a maximum weight matching (M_{OPT}) of G. Let M_{LGB} be the matching corresponding to a LGB allocation at a price vector P^*.

Consider the difference graph $G_D = (A, B, E_D, W)$ where $E_D = (M_{LGB} \cup M_{OPT}) - (M_{LGB} \cap M_{OPT})$. Every path in G_D consists of alternating edges from M_{LGB} and M_{OPT}. Every item will be sold (matched) in both the allocations (because of dummy bidders). Therefore every maximal path in G_D starts with a bidder vertex and ends at a bidder vertex. Define weight of a subset of edges $Q \subset E_D$ in G_D as:

$$W(Q) = \sum_{e \in Q \ \& \ e \in M_{OPT}} W(e) - \sum_{e \in Q \ \& \ e \in M_{LGB}} W(e).$$

Clearly $(M_{OPT}) = W(E_D) + W(M_{LGB})$.

Claim 1 *No cycle Q in G_D with q bidders has a weight greater than ϵq.*

Proof: Without loss of generality, consider a cycle Q as shown in Figure 1. The dashed edges represent the LGB allocations and the solid edges represent the optimal allocation. Since dummy bidders have valuations on only one item, none of the bidders in Q can be dummy bidders. Since bidder 1 places a winning bid on item q under LGB, its surplus on item q would have been the largest. Let the highest bid on item 1 at that instant be p_1, then

$$v_{1q} - (p_q^* - \epsilon) \geq v_{11} - p_1.$$

Since the final price of item 1 can only rise we have:

$$v_{1q} - (p_q^* - \epsilon) \geq v_{11} - p_1^*.$$

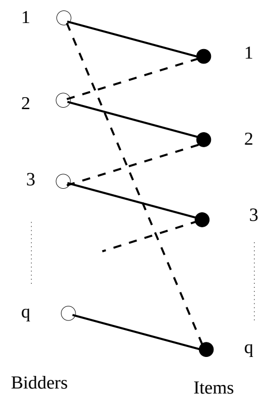

Figure 2: A cycle

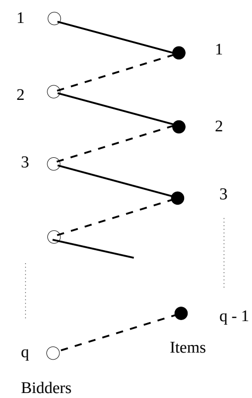

Figure 3: A path

Similarly for bidder i, $(i \geq 2)$ we have:

$$v_{ii-1} - (p_{i-1}^* - \epsilon) \geq v_{ii} - p_i^*.$$

Adding the above equations for all values of i yields:

$$W(Q) \leq \epsilon q.$$

□

Claim 2 *No maximal path Q of G_D, with q bidders has weight greater than ϵq.*

Proof: Consider a maximal path Q of G_D. WLOG consider the path shown in Figure 2. Since bidder 1 didn't get any item in LGB,

$$v_{11} \leq p_1^* + \epsilon.$$

Since bidder i $(i \neq q)$ preferred item $i - 1$ over item i in the LGB bid, we have

$$v_{ii} - p_i^* \leq v_{ii-1} - (p_{i-1}^* - \epsilon).$$

Since bidder q bids for item $q-1$ in LGB, its valuation of item $q-1$ must be greater than or equal to the price at which it was sold:

$$v_{qq-1} \geq p_q^*.$$

Note that the above equation holds even if bidder i was a dummy bidder. Adding above equations over i yields:

$$W(Q) \leq \epsilon(q-3).$$

□

Since graph G_D is difference of two matchings, it can be decomposed into disjoint paths and cycles. From Claim 1 and 2 it follows that $W(E_D) \leq m\epsilon$. Since $W(E_D)$ can also be written as $W(M_{OPT}) - W(M_{LGB})$, therefore $W(M_{OPT}) - W(M_{LGB}) \leq m\epsilon$. □

Proof: [Lemma 2] Order the items of C in ascending order of the time at which the winning bid on each item was received. Consider the item j which is last in this sequence and consider the system at the time when all the bids on all items in C except the winning bid on this item had been received. Since for all items $j' \neq j, j' \in C$, the eventual winners had their bids outstanding at this time, they could not have simultaneously had an outstanding bid on j. If there already existed some bid on j at this time, then the bidder who placed that bid did not get any item from C in the final allocation. If there did not exist any bid on j at this time, the item j receives only one bid, that of its dummy bidder (auctioneer). Since all bids by dummy bidders precede any bid by an exogenous bidder, therefore no item in C got any bids from any exogenous bidder. □

Proof: [Lemma 3] Assume, for contradiction that bidder l is not outbid. Let P^2 be the price vector corresponding to this outcome and M_2 be the corresponding allocation (matching). Since bidder l's bid on item j is not outbid, we have $p_j^2 = p_j$. Let M_1 be the allocation (matching) in the outcome when all bidders follow LGB.

In order to prove Lemma 3 we need to establish certain properties. Let Q be a cycle in the graph $G' = (A, B, M_1 \cup M_2)$. The following claim says that if price of one item in Q falls (in the second outcome) by a significant amount, then the prices of all the items in Q will also fall by similar amounts.

Claim 3 *If items $j, k \in Q$ and $p_j^1 - p_j^2 = \delta \geq 0$ then $p_k^1 - p_k^2 \geq \delta - 2(q-1)\epsilon$, where q is number of items in Q.*

Proof: Suppose bidder i is the winner of item j in the first outcome and bidder i gets another item j' in the second outcome. When bidder i placed a bid on item j' in the second outcome, it had the option of placing a bid on item j. Let the current winning bid on item j at that time be of amount p_j. Therefore from the property of LGB bidding for bidder i:

$$v_{ij'} - (p_{j'}^2 - \epsilon) \geq v_{ij} - p_j.$$

Since the prices can only increase, the final price p_j^2 of item j will be at least p_j. Therefore:

$$v_{ij'} - (p_{j'}^2 - \epsilon) \geq v_{ij} - p_j^2.$$

Using a similar argument for the first outcome, we have:

$$v_{ij} - (p_j^1 - \epsilon) \geq v_{ij'} - p_{j'}^1.$$

The above two equations yield:

$$p_{j'}^1 - p_{j'}^2 \geq p_j^1 - p_j^2 - 2\epsilon$$

or

$$p_{j'}^1 - p_{j'}^2 \geq \delta - 2\epsilon.$$

Applying the same argument to winner of item j' in the first outcome, the fall in the price of the next item in the cycle Q can be bounded. Since there are $q-1$ items in the cycle, excluding item j, the minimum fall in price of any item $k \in Q$ can be bounded as:

$$p_k^1 - p_k^2 \geq \delta - 2q\epsilon.$$

□

Note that the above claim is not restricted to the cycle containing the item j. It may be applied to any cycle.

Claim 4 *Let bidder i be a marginal loser of a cycle Q of G', in the first outcome, and gets item h in the second outcome. If price of an item falls by δ then price of item h will fall by at least $\delta - (2q+1)\epsilon$ in the second outcome.*

Proof: Assume that bidder i places a second highest bid on item j' of Q in the first outcome. By Claim 3, we have:

$$p_{j'}^1 - p_{j'}^2 \geq \delta - 2(q-1)\epsilon.$$

Now consider the following two cases:

Case 1: Bidder i wins item k in the first outcome. When the bidder placed the second highest bid on item j', it had more surplus on item j' than on item k. So we have:

$$v_{ij'} - (p_{j'}^1 - 2\epsilon) \geq v_{ik} - (p_k^1 - \epsilon). \tag{1}$$

Since the bidder choose item k over item h in the first outcome, we have:

$$v_{ik} - (p_k^1 - \epsilon) \geq v_{ih} - p_h^1. \tag{2}$$

In the second outcome, the bidder choose item h over item j', so we have:

$$v_{ih} - (p_h^2 - \epsilon) \geq v_{ij'} - p_{j'}^2.$$

Adding above four equations yields:

$$p_h^1 - p_h^2 \geq \delta - (2q+1)\epsilon. \tag{3}$$

Case 2: Bidder i wins no item in the first outcome. When bidder i placed a bid on item j', it had a non negative surplus on it:

$$v_{ij'} - (p^1_{j'} - \epsilon) \geq 0.$$

Since bidder i did not bid on item h in the first outcome, its surplus on item h would have been negative:

$$0 \geq v_{ih} - (p^1_h + \epsilon).$$

Combining above two equations with Eq. 8 and 8 yields Eq. 3. □

We now continue the proof for Lemma 3. From our assumption we have, $p^1_j - p^2_j > 3m\epsilon$.

Claim 5 *The maximal path containing item j in graph $G' = (A, B, M_1 \cup M_2)$ is a cycle.*

Proof: If the maximal path is not a cycle, it must end at a bidder i who gets an item j' in the first outcome and does not get any item in the second outcome. On the lines of Claim 3, it can be argued that the price of item j' in the second outcome falls by some non zero amount. Therefore, bidder i should have a positive surplus on this item and should have placed a bid on this item in the second outcome, which is contradiction to our assumption. □

Claim 6 *Every cycle in graph $G' = (A, B, M_1 \cup M_2)$ has a marginal loser in the first outcome.*

Proof: Let Q be a cycle of G'. Since dummy bidders have valuations on only one item, no dummy can be in a cycle. If there is no marginal loser of Q in the first outcome, then by Lemma 2, there is an item in Q which is sold at the reserve price in the first outcome. Therefore, the dummy bidder corresponding to that item is in Q which is impossible. □

Claim 7 *If bidder i is a marginal loser, in the first outcome, of the cycle containing item j, then bidder i gets an item in the second outcome.*

Proof: Let Q be the cycle of G' containing the item j. Assume for contradiction that bidder i does not get any item in the second outcome. By Claim 3 and our assumption that $p^1_j - p^2_j > 3m\epsilon$, prices of all the items in Q must fall by more than 3ϵ. Therefore the marginal loser i should have a non negative surplus on an item in Q and should place a bid on an item in Q in the second outcome. This is a contradiction to our assumption. Therefore the marginal loser of Q in the first outcome should get an item in the second outcome. □

We expand the set Q by adding the cycle corresponding to the marginal loser of Q in the first outcome. Using above two claims and Claim 3 and 4 it follows that the price of every item in the new set Q, falls by more than $\delta - 3(q-1)\epsilon$. Finally, a set Q is obtained whose marginal loser in the first outcome (say bidder i) does not get an item in the second outcome. Bidder i was a marginal loser in Q in the first outcome and prices of items in Q drop by an amount greater than 3ϵ in the second outcome. Therefore bidder i should have non-negative surplus on at least one item in Q in the second outcome, which is impossible.

So, our assumption that bidder l is not outbid in the final outcome of the LGB must be wrong. Hence the proof of Lemma 3. □

Proof: [Theorem 3] Assume that a bidder i follows any strategy and all other bidders follow LGB. Let P^* be the final price vector if bidder i was also to follow LGB. It is clear from Lemma 3 that the bidder i cannot win any item j at a price less than $p^*_j - 3m\epsilon$. If it did not win any item when LGB was used (and so had a zero surplus on all items at the price vector P^*), it cannot get a surplus of greater than $3m\epsilon$ by winning any item using any other strategy. On the other hand, if the bidder i had won an item j when LGB was used, it cannot increase its surplus by more than $3m\epsilon$ by winning the same item j. Further, since item j was the surplus maximizing item for bidder i at P^*, the bidder cannot increase its surplus by more than $3m\epsilon$ by winning any other item when prices do not fall by more than $3m\epsilon$. This completes the proof. □